A Multi-Level Approach

To

Context-preserving Smooth Function Extension

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Contents

Heat diffusion PDE models

Isotropic diffusion

(constant conductivity)

$$\begin{cases} \dfrac{\partial}{\partial t} u(x,t) = \mathrm{Div}\left(c \nabla u(x,t)\right) \\ u(x,0) = u_0(x) \end{cases}$$

Anisotopic diffusions

(non-constant conductivity)

$$\begin{cases} \dfrac{\partial}{\partial t} u(\bullet,t) = \mathrm{Div}\left(c(f(u),\bullet,t) \nabla u(\bullet,t)\right) \\ u(x,0) = u_0(x) \end{cases}$$

Perona-Malik anisotropic diffusion PDE

c as a function of only one variable

Perona and Malik 1990 $u_t = \text{Div}(c(|\nabla u|)\nabla u)$

$$\begin{cases} u_t = \text{Div}(c(|\nabla u|)\nabla u) & \text{in } D \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial D \\ u(x,0) = u_0(x) & \text{in } D \end{cases}$$

where D is the image domain

Refomulation of anisotropic diffusion

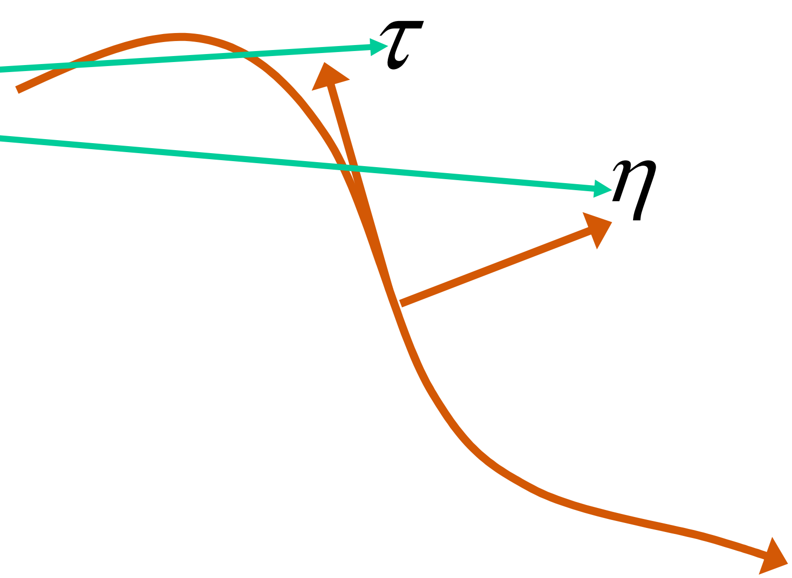
(in terms of local coordinates)

$$\eta = \nabla u / |\nabla u|, \quad \phi(p) = p\,c(p), \; p = |\nabla u|.$$

Diffusions along normal and tangential directions

$$\phi'(p) = c(p) + p c'(p)$$

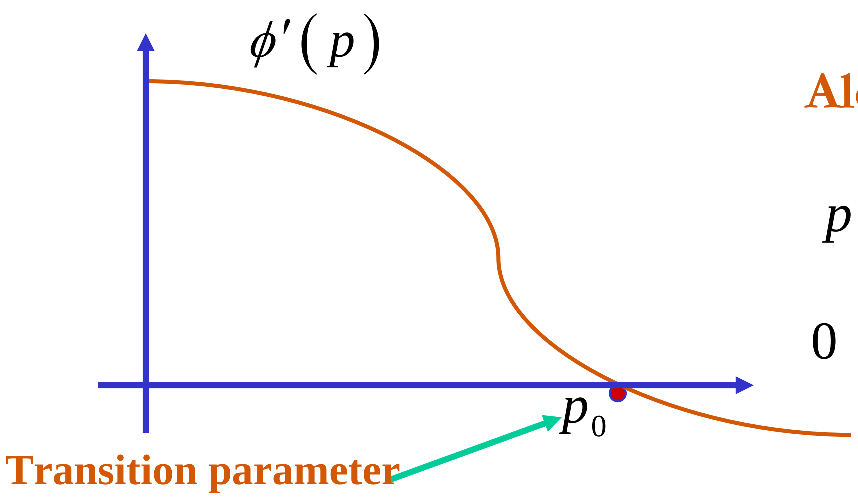
$$\begin{cases} u_t = \boxed{\phi'(p) u_{\eta\eta}} + \boxed{c(p) u_{\tau\tau}} \\ \dfrac{\partial u}{\partial n} = 0 \quad \text{on } \partial D \\ u(x,0) = u_0(x) \quad \text{in } D \end{cases}$$



Perona-Malik condition

$$c(p) > 0 \text{ decreases to 0 on } [0, \infty)$$

$$\phi'(p) = c(p) + pc'(p) \rightarrow \begin{cases} \phi'(p) > 0, & \text{for } 0 \le p < p_0 \\ \phi'(p) < 0, & \text{for } p > p_0 \end{cases}$$



Along the normal direction :

$p > p_0$ **Backward diffusion**

$0 \le p < p_0$ **Forward diffusion**

Conductivity functions

$$c(p) = e^{-\frac{p^2}{2p_0^2}}$$ **Gaussian**

$$c(p) = \frac{p_0^2}{p_0^2 + p^2}$$ **Lorentz**

$$c(p) = \min(1, \frac{1}{p})$$ **Normalized TV**

$$c(p) = 1$$ **Isotropic**

p_0 **transition parameter**

$p > p_0$ **for backward diffusion**

Anisotropic diffusion preserves image edges

Target (unknown)

Input image

Isotropic

Anisotropic

Lagged diffusion Model

For $j = 0, 1, \ldots,$ let $u^{j+1} = u^{j+1}(x,t)$

be the solution of

$$
\begin{cases}
u_t^{j+1} = \mathrm{Div}(c(|\nabla u^j|)\nabla u^{j+1}) & \text{in } D \\
\dfrac{\partial u^{j+1}}{\partial \mathbf{n}} = 0 & \text{on } \partial D \\
u^{j+1}(x,0) = u_0(x) & \text{in } D
\end{cases}
$$

with input $u^0(x,t) = u_0(x)$

Lagged diffusion model assures good approximation

Assuming "smooth" input u_0, the convergence of $u^j = u^j(x,t)$ to the solution $u = u(x,t)$ of

$$\begin{cases} u_t = \mathrm{Div}(c(|\nabla u|)\nabla u) \quad \text{in } D \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0 \quad \text{on } \partial D \\ u(x,0) = u_0(x) \quad \text{in } D \end{cases}$$

is uniform on D x $[0, T]$ for any $T > 0$, with (small "oh" of) geometric convergence rate.

Contents

Problem statement

Simply connected domain Ω with holes $D_1, \cdots, D_m$ in $\mathbb{R}^d$, $d > 1$. Consider the "Swiss cheese" domain

$$\tilde{\Omega} := \Omega \setminus (D_1 \cup \cdots \cup D_m)$$

Given a sufficiently smooth function on Ω, but with missing portion on $D_1 \cup \cdots \cup D_m$. The problem is to recover the missing portion.

Since our approach is local. it is sufficient to consider a single hole $D \subset \Omega$ so that $\tilde{\Omega} := \Omega \setminus D$

Diffusion and data propagation

Multi-level heat kernels: $G_j(x,y)$

Diffusion operators: $$(T_j f)(x) = \int_D f(y) G_j(x,y)\,dy$$

Data propagation operators: $$P_j(d_j F)(x) = -\int_{\partial D} (d_j F)(y) g_j(x,y)\,ds$$

Detail-extension operators: $$E_j = T_0 \cdots T_{j-1} P_j, \quad j = 1, \cdots, n$$

Wavelet details: $$w_j = E_j(d_j(F)), \quad j = 1, \ldots, n$$

Lagged anisotropic conductivity

Let

$$c_j(x) := c(|\nabla u^j(x)|) \quad \text{or} \quad c(|\nabla s * u^j(x)|),\ x \in D \subset \mathbb{R}^d$$

for some lowpass function ***s***, such as the Gaussian,

with

$$c_{-1}(x) = 1$$

and consider the differential operators

$$(L_j f)(x) = \nabla \cdot \left(c_{j-1}(x) \nabla f(x)\right), \quad x \in D$$

Diffusion, propagation, and detail-extension

Green's function:

$$\begin{cases} (L_j)G_j(x,y) & = \delta(x,y), \quad x,y \in D \\ G_j(x,y)\Big|_{y\in\partial D} & = 0, \quad x \in D \end{cases}$$

"Diffusion" operator:

$$(T_j f)(x) = \int_D f(y)G_j(x,y)dy \qquad (1)$$

Propagation kernel:

$$g_j(x,y) := -c_{j-1}(y)\left(\frac{\partial}{\partial \mathbf{N}} G_j(x,\cdot)\right)(y) \qquad (2)$$

Propagation operator:

$$(P_j v)(x) = \int_{\partial D} v(y(s))g_j(x,y(s))ds \qquad (3)$$

Detail-extension operator:

$$E_j = T_0 \cdots T_{j-1}P_j, \quad j = 1, \cdots, n \qquad (4)$$

The ML approach

1. Apply the propagation operator P_0 defined in (3) with $j = 0$ to transport the boundary data $d_0(F) = F|_{\partial D}$ to D to construct the lowest resolution level (called thumb-nail).
2. Compute mixed derivatives $d_1(F), \cdots, d_n(F)$ of of increasing order, using data $F(x)$ from the exterior of D.
3. Extend the mixed derivatives data $d_1(F), \cdots, d_n(F)$ by applying the operators $E_1, \ldots, E_n$ defined in (4) to D.

Contents

Extension of Green's formula

Lemma 1. *For sufficiently smooth functions* $f, h, c_{j-1} \in C(D)$ *with* $h|_{\partial D} = 0$,

$$\int_D f(y)(L_j h)(y)dy = \int_D h(y)(L_j f)(y)dy - \int_{\partial D} c_{j-1}(y)f(y)\frac{\partial}{\partial \mathbf{N}} h(y)ds,$$

where y *is parametrized in terms of* s.

Boundary data, wavelets, and MRA solution

Choice of data: $d_j(F) := L_{j-1} \cdots L_0 F \quad \text{on} \quad \partial D$

Lowest resolution level: $u_0 = P_0(F\big|_{\partial D})$

Wavelet details: $w_j = E_j(d_j(F)), \quad j = 1, \ldots, n$

MRA solution: $u_n := u_0 + w_1 + \cdots + w_n$

Continuity preservation: $u_n\big|_{\partial D} = u_0\big|_{\partial D} = F\big|_{\partial D}, \quad n = 1, 2, \ldots$

Error formula

Theorem 1 (ACHA, 08) ***Let F be a sufficiently smooth function in*** Ω
but with missing portion $F_D := F\big|_D, \ D \subset \Omega$

Then the error of ML recovery of the lost data is given by

$$F_D(x) - u_n(x) = (T_0 \cdots T_n)(L_n \cdots L_0 F_D)(x), \quad x \in D.$$

Error bound

In particular,

$$\left\|F_D - u_n\right\|_D \leq C_n \prod_{j=0}^{n} \left\|T_j\right\|,$$

where

$$C_n := \left\|(L_n \cdots L_0) F_D\right\|_D$$

and

$$\left\|T_j\right\| := \sup_{x \in D} \int_D |G_j(x,y)| dy.$$

Proof: Lowest resolution

As a consequence of Lemma 1,

$$\begin{aligned} F_D(x) &= \int_D F_D(y) L_0 G_0(x,y)\,dy \\ &= \int_D (L_0 F_0)(y) G_0(x,y)\,dy - \int_{\partial D} F(y) c_{-1}(y) \frac{\partial}{\partial \mathbf{N}} G_0(x,y)\,ds \\ &= (T_0(L_0 F_D))(x) + u_0(x) \end{aligned}$$

or

$$F_D(x) - u_0(x) = (T_0(L_0 F_D))(x)$$

Proof: Higher resolutions

For each *j =1*, …, *n*, an application of Lemma 1 also yields:

$$\begin{aligned} f_j(x) :&= (L_{j-1}\cdots L_0 F_D)(x) = \int_D f_j(y) L_j G_j(x,y)\,dy \\ &= \int_D (L_j f_j)(y) G_j(x,y)\,dy - \int_{\partial D} f_j(y) g_j(x,y)\,ds \\ &= (T_j(L_j \cdots L_0 F_D))(x) - \int_{\partial D} (d_j F)(y) g_j(x,y)\,ds \\ &= (T_j(L_j \cdots L_0 F_D))(x) + P_j(d_j F)(x). \end{aligned}$$

Proof: Wavelet details

Hence, by applying the operator $T_0 \cdots T_{j-1}$ to each term and the definition (4), we have

$$(T_0 \cdots T_{j-1})(L_{j-1} \cdots L_0 F_D)(x) = (T_0 \cdots T_j)(L_j \cdots L_0 F_D)(x) + (E_j(d_j F))(x),$$

which, by telescoping, yields

$$(T_0(L_0 F_D))(x) - \sum_{j=1}^{n}(E_j(d_j F))(x) = (T_0 \cdots T_n)(L_n \cdots L_0 F_D)(x).$$

Recall that

$$F_D(x) - u_0(x) = (T_0(L_0 F_D))(x),$$

$$w_j = E_j(d_j(F)), \quad j = 1, \ldots, n$$

Extension to Manifolds

In a joint paper with H. Mhaskar (ACHA, 09)

Smooth function extension on manifolds with context recovery

(not discussed in this talk)

Contents

Special case: Isotropic setting

Conductivity: $c(p) = 1$

Differential operators: $L_j = \triangle$

Boundary data: $d_0(F) = F\big|_{\partial D}$ and $d_j(F) = \triangle^j F\big|_{\partial D}, \quad j = 1, \ldots, n$

Wavelets: $w_j = E_j(d_j(F)), \quad j = 1, \ldots, n$

MRA solution: $u_n := u_0 + w_1 + \cdots + w_n$

Isotropic setting: Peano kernels

Lemma 2. The "Peano" kernels

$$K_1(x,y) := G^{\triangle}(x,y)$$

$$K_{j+1}(x,y) := \int_{D^j} G^{\triangle}(x,y_j)\cdots G^{\triangle}(x_k,y_{k-1})\cdots G^{\triangle}(y_1,y)dy_1\cdots dy_j$$

$j = 1, 2, \ldots, n,$ satisfy:

$$\triangle_x K_{j+1}(x,y) = K_j(x,y)$$

$$\triangle_x^j K_j(x,y) = \delta(x,y)$$

Isotropic setting: Formulas

Propagation operator: $(Pf)(x) := \int_{\partial D} f(y) \frac{\partial}{\partial \mathbf{N}} G^{\triangle}(x,y)\, ds$

Ground level: $u_0(x) = P(F|_{\partial D})(x)$

Wavelets: $w_j(x) = \int_D K_j(x,y) P(\triangle^j F|_{\partial D})(y)\, dy$

MRA: $u_n(x) = u_0(x) + w_1(x) + \cdots + w_n(x)$

Error (Remainder): $F(x) - u_n(x) = \int_D K_{n+1}(x,y) \triangle^{n+1} F(y)\, dy$

Isotropic setting: Interpolation properties

Theorem 2. For $k = 0, \ldots, n$ and $j = 1, \ldots, n$, the wavelet components satisfy the boundary conditions:

$$\triangle^k w_j \Big|_{\partial D} = \delta_{j,k} \ \triangle^j F \Big|_{\partial D}$$

Hence, the continuous function extension

$$u_n(x) = u_0(x) + w_1(x) + \cdots + w_n(x)$$

satisfies the interpolating property:

$$u_n \Big|_{\partial D} = F \Big|_{\partial D} \qquad \triangle^j u_n \Big|_{\partial D} = \triangle^j F \Big|_{\partial D}, \quad j = 1, \ldots, n$$

Smooth harmonic inpainting

T. Chan and J. Shen

Dirichlet problem:
$$\begin{cases} \triangle u^h = 0 \quad \text{in} \quad D, \\ u^h\big|_{\partial D} = F\big|_{\partial D} \quad \text{on} \quad \partial D \end{cases}$$

Linear inpainting:
$$\left\| F_D - u^h \right\|_D = O(\epsilon^2)$$

Cubic inpainting:
$$\left\| F_D - u^{bh} \right\|_D = O(\epsilon^4)$$

Error parameter:
$$\text{diam } (D) = 2\epsilon > 0$$

Smooth bi-harmonic inpainting

T. Chan and J. Shen

Bi-harmonic inpainting function:

$$u^{bh} := u^h + u^a$$

Detail component:

$$\begin{cases} \triangle u^a = \tilde{u}^h & \text{in } D, \\ u^a\big|_{\partial D} = 0 & \end{cases}$$

where

$$\begin{cases} \triangle \tilde{u}^h = 0 & \text{in } D, \\ \tilde{u}^h\big|_{\partial D} = \triangle F\big|_{\partial D} & \text{on } \partial D \end{cases}$$

MRA extension and sharp error estimates

Linear inpainting: $u^h = u_0$

Cubic inpainting: $u^{bh} = u^h + u^a = u_0 + w_1$

Our continuous function extension is valid for any number of levels $u_n := u_0 + w_1 + \cdots + w_n$

From Theorem 1, we also have sharp error estimates:

$$\left\|F_D - u_n\right\|_D \leq \left\|\Delta^{n+1} F_D\right\|_D \left\|T^\Delta\right\|^{n+1}$$

Improvement and extension of Chan-Shen error estimates

Error parameter: $\operatorname{diam}(D) = 2\epsilon > 0$

Green's function of Laplacian: $\|T^{\triangle}\| := \sup_{x \in D} \int_D |G^{\triangle}(x,y)dy| = \sup_{x \in D} \left| \int_D G^{\triangle}(x,y)dy \right|$

For $D = D_\epsilon := \{|x - x_0| < \epsilon\} \subset \mathbb{R}^d$

$$\int_D G^{\triangle}(x,y)dy = \frac{|x - x_0|^2 - \epsilon^2}{2d}$$

Hence,

$$\|F_D - u_n\|_D \le \|\Delta^{n+1} F_D\|_D \left(\frac{\epsilon^2}{2d}\right)^{n+1} \quad n = 0, 1, \ldots$$

This holds for $D \subset D_\epsilon$:

Two-dimensional setting

Green's function: $G^{\triangle}(x,y) = \frac{1}{2\pi}\, ln\, |\phi_x(y)|$

where $\phi_x(y), x \in D \subset \mathbb{R}^2$ is the conformal map of *D* onto the unit disk with center at **0**, such that $\phi_x(x) = 0$ and $\phi'_x(x) > 0$ so that

$$\int_D G^{\triangle}(x,y)dy$$

can be estimated more directly.

Thank You

Questions and comments?