A quantitive notion of redundancy for finite frames

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Definition of a Frame

1952 Duffin/Schaeffer

$(\varphi_i)_{i=1}^N$ is a **frame** for a n-dimensional Hilbert space $\mathbb{H}_n$ if there exists (*lower, upper frame bounds*) $A, B > 0$ so that

$$A\|x\|^2 \leq \sum_{i=1}^{N} \langle x, \varphi_i \rangle|^2 \leq B\|x\|^2 \quad \forall x \in H$$

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In this case we say the frame has **redundancy** $\frac{N}{n}$.

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We would like a notion of **frame redundancy** to distinguish between our three examples above. To achieve this goal, for a frame Φ we will introduce:

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Upper Redundancy $\mathcal{R}_{\Phi}^{+}$

Lower Redunancy $\mathcal{R}_{\Phi}^{-}$

Desiderata 1

Triviality. Redundancy should not count zero vectors.

Desiderata 2

Generalization. If Φ is an equal-norm Parseval frame, then in this special case the customary notion of redundancy shall be attained, i.e., $\mathcal{R}_\Phi^- = \mathcal{R}_\Phi^+ = \frac{N}{n}$.

Desiderata 3

Nyquist Property. The condition $\mathcal{R}_{\Phi}^{-} = \mathcal{R}_{\Phi}^{+}$ shall characterize tightness of a normalized version of Φ, thereby supporting the intuition that upper and lower redundancy being different implies 'non-uniformity' of the frame. In particular, $\mathcal{R}_{\Phi}^{-} = \mathcal{R}_{\Phi}^{+} = 1$ shall be equivalent to orthogonality as the 'limit-case'.

Desiderata 4

Upper and Lower Redundancy. Upper and lower redundancy shall be 'naturally' related by $0 < \mathcal{R}_{\Phi}^{-} \leq \mathcal{R}_{\Phi}^{+} < \infty$.

Desiderata 5

Additivity. Upper and lower redundancy shall be subadditive and superadditive, respectively, with respect to unions of frames. They shall be additive provided that the redundancy is uniform, i.e., $\mathcal{R}_\Phi^- = \mathcal{R}_\Phi^+$.

Desiderata 6

Invariance. Redundancy shall be invariant under the action of a unitary operator on the frame vectors, under scaling of single frame vectors, as well as under permutation, since intuitively all these actions should have no effect on, for instance, robustness against erasures, - which is one property redundancy shall intuitively measure.

Desiderata 7

Spanning Sets. The lower redundancy shall measure the maximal number of spanning sets of which the frame consists. This immediately implies that the lower redundancy is a measure for robustness of the frame against erasures in the sense that any set of a particular number of vectors can be deleted yet leave a frame.

Desiderata 8

Linearly Independent Sets. The upper redundancy shall measure the minimal number of linearly independent sets the frame consists of.

Local Redundancy

Notation: If $y \in \mathcal{H}_n$, $< y >$ will denote the span of y and $P_{<y>}$ will denote the orthogonal projection onto $< y >$. The unit sphere of $\mathcal{H}_n$ will be denoted $\mathbb{S}_n$.

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Definition

Let $\Phi = (\varphi_i)_{i=1}^N$ be a frame for a finite-dimensional real or complex Hilbert space $\mathcal{H}_n$. For each $x \in \mathbb{S}_n$, the *redundancy function* $\mathcal{R}_\Phi : \mathbb{S}_n \to \mathbb{R}^+$ is defined by

$$\mathcal{R}_\Phi(x) = \sum_{i=1}^N \|P_{\langle \varphi_i \rangle}(x)\|^2.$$

Equivalently

An equivalent formulation of $\mathcal{R}_{\Phi}(x)$ is:

$$\mathcal{R}_{\Phi}(x) = \sum_{i:\phi_i \neq 0} \left| \left\langle x, \frac{\phi_i}{\|\phi_i\|} \right\rangle \right|^2$$

A New Notion of Redundancy

Definition

Let $\Phi = (\varphi_i)_{i=1}^N$ be a frame for a finite-dimensional real or complex Hilbert space $\mathcal{H}_n$. Then the *upper redundancy of Φ* is defined by

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and the *lower redundancy of Φ* by

$$\mathcal{R}_\Phi^- = \min_{x \in \mathbb{S}_n} \mathcal{R}_\Phi(x).$$

Moreover, Φ has a *uniform redundancy*, if

$$\mathcal{R}_\Phi^- = \mathcal{R}_\Phi^+.$$

Another Formulation

Note that this definition yields:

$$\mathcal{R}_\Phi^- = A \qquad \mathcal{R}_\Phi^+ = B$$

where A, B are the lower and upper frame bounds of

$$\{\frac{\varphi_i}{\|\varphi_i\|} : \varphi_i \neq 0\}$$

Desiderata Revisted

This definition satisfies all eight of our Desiderata for redundancy of frames. In particular, Let $\Phi = (\varphi_i)_{i=1}^N$ be a frame for an n-dimensional real or complex Hilbert space $\mathcal{H}_n$. Then we have:

Generalization

If Φ is an equal-norm Parseval frame, then

$$\mathcal{R}_\Phi^- = \mathcal{R}_\Phi^+ = \frac{N}{n}.$$

Nyquist Property

The following conditions are equivalent:

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Also the following conditions are equivalent:

(i') We have $\mathcal{R}_{\Phi}^{-} = \mathcal{R}_{\Phi}^{+} = 1$.

(ii') Φ is orthogonal.

Upper and Lower Redundancy

We have

$$0 < \mathcal{R}_{\Phi}^{-} \leq \mathcal{R}_{\Phi}^{+} < \infty.$$

Additivity

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$$\mathcal{R}^{\pm}_{\Phi \cup (e_i)_{i=1}^n} = \mathcal{R}^{\pm}_{\Phi} + 1.$$

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Moreover, for each frame Φ' in $\mathcal{H}_n$,

$$\mathcal{R}^{-}_{\Phi \cup \Phi'} \geq \mathcal{R}^{-}_{\Phi} + \mathcal{R}^{-}_{\Phi'} \quad \text{and} \quad \mathcal{R}^{+}_{\Phi \cup \Phi'} \leq \mathcal{R}^{+}_{\Phi} + \mathcal{R}^{+}_{\Phi'}.$$

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In particular, if Φ and Φ' have uniform redundancy, then

$$\mathcal{R}^{-}_{\Phi\cup\Phi'} = \mathcal{R}_{\Phi} + \mathcal{R}_{\Phi'} = \mathcal{R}^{+}_{\Phi\cup\Phi'}.$$

Invariance

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and under permutations, i.e.,

$$\mathcal{R}^{\pm}_{(\varphi_{\pi(i)})_{i=1}^N} = \mathcal{R}^{\pm}_{\Phi}, \quad \pi \in S_{\{1,\ldots,N\}},$$

Spanning Sets

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In particular, any set of $\lfloor \mathcal{R}_{\Phi}^{-} \rfloor - 1$ vectors can be deleted yet leave a frame.

Linearly Independent Sets

If Φ does not contain any zero vectors, then it can be partitioned into $\lceil \mathcal{R}_\Phi^+ \rceil$ linearly independent sets.

Our Example 1

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Then

$$\mathcal{R}_{\Phi_1}^- = 2 = \mathcal{R}_{\Phi_1}^+,$$

and this is an equal norm tight frame.

Our Example 2

$$\Phi_2 = (e_i)_{i=1}^n \cup \{e_1, e_1, \ldots, e_1\} \text{ where } e_1 \text{ occurs } n\text{-times}$$

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$$\mathcal{R}^-_{\Phi_2} = 1 \quad \mathcal{R}^+_{\Phi_2} = n + 1$$

Moreover,

any partition of Φ_2 contains at most one spanning set,

any partition of Φ_2 into linearly independent sets must contain at least $(n+1)$-sets.

Our Example 3

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$$\Phi_3 = (e_i)_{i=1}^n \cup \{0, 0, \ldots, 0\} \text{ where 0 occurs } n\text{-times}$$

Then

$$\mathcal{R}^-_{\Phi_3} = \mathcal{R}^+_{\Phi_3} = 1$$

Moreover,

Φ_3 contains one spanning set and one linearly independent set.

Bounds on Upper and Lower Redundancy

Theorem

If the $\{\varphi_i\}$ *are non-zero vectors, then the upper and lower redundancies of a frame* $\Phi = (\varphi_i)_{i=1}^N$ *for a real or complex Hilbert space* $\mathcal{H}_n$ *having dimension* $n \geq 2$ *satisfy the inequalities*

$$0 < \mathcal{R}_\Phi^- \leq \frac{N}{n} \leq \mathcal{R}_\Phi^+ < N,$$

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$$0 < \mathcal{R}_\Phi^- \leq \frac{N}{n} \leq \mathcal{R}_\Phi^+ < N,$$

Moreover, if $\mathcal{R}_\Phi^- = \frac{N}{n}$ or $\mathcal{R}_\Phi^+ = \frac{N}{n}$, then the normalized version of Φ is a tight frame.

Occurance of Upper and Lower Redundancy

Theorem

Let $n \leq N$, $r_1 \in (0, \frac{N}{n}]$, and $r_2 \in [\frac{N}{n}, N)$. Then the following conditions are equivalent.

(i) *There exists a frame $\Phi = (\varphi_i)_{i=1}^N$ for $\mathcal{H}_n$, $n \geq 2$, such that*

$$\mathcal{R}_\Phi^- = r_1 \quad \text{and} \quad \mathcal{R}_\Phi^+ = r_2.$$

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(ii) *We have*

$$(n-1)r_1 + r_2 \leq N.$$

Constructing Frames with given Redundancies

Theorem

For every $r_1 \in (0, \frac{N}{n}]$ *and every* $r_2 \in [\frac{N}{n}, N)$*, we can find unit-norm frames* $\Phi = (\varphi_i)_{i=1}^N$ *and* $\Psi = (\psi_i)_{i=1}^N$ *with*

$$\mathcal{R}_\Phi^- = r_1 \quad \textit{and} \quad \mathcal{R}_\Psi^+ = r_2.$$

Characterization of Redundancy Functions

Theorem

Let $f : \mathbb{S}_n \to \mathbb{R}_0^+$, $\mathcal{H}$ *be an n-dimensional real or complex Hilbert space with* $n \geq 3$, *and let* q *be the extension of* f *to* $\mathcal{H}$ *given by* $q(0) = 0$ *and* $q(x) = \|x\|^2 f(x/\|x\|)$ *for any* $x \neq 0$. *Let* ω *denote the probability measure on the unit sphere which is invariant under all unitaries. Then the following conditions are equivalent:*

Characterization of Redundancy Functions

Theorem

(i) *There exists a frame* Φ *for* $\mathcal{H}_n$ *such that*

$$f(x) = \mathcal{R}_\Phi(x) \quad \text{for all } x \in \mathbb{S}_n.$$

Characterization of Redundancy Functions

Theorem

(i) *There exists a frame Φ for $\mathcal{H}_n$ such that*

$$f(x) = \mathcal{R}_\Phi(x) \quad \textit{for all } x \in \mathbb{S}_n.$$

(ii) *The function f is strictly positive on $\mathbb{S}$, its extension q satisfies the parallelogram identity*

$$q(x+y) + q(x-y) = 2(q(x) + q(y)) \quad \textit{for all } x, y \in \mathbb{S}_n$$

and f integrates to

$$\int_{\mathbb{S}_n} f(x) d\omega(x) = N/n$$

with some integer $N \geq n$.

The extreme cases

Let $0 < \varepsilon < 1$ and choose $\Phi_3 = (\varphi_i)_{i=1}^N$ as

$$\varphi_i = \begin{cases} e_1 & : \quad i = 1, \\ \sqrt{1-\varepsilon^2}e_1 + \varepsilon e_i & : \quad i \neq 1. \end{cases}$$

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Also, this frame is a linearly independent set.

Redundancy for this example

$$\begin{aligned}
\mathcal{R}_{\Phi_3}(e_1) &= \sum_{i=1}^{N} \|P_{\langle \varphi_i \rangle}(e_1)\|^2 \\
&= 1 + \sum_{i=2}^{N} |\langle e_1, \sqrt{1-\varepsilon^2}e_1 + \varepsilon e_i \rangle|^2 \\
&= 1 + (N-1)(1-\varepsilon^2).
\end{aligned}$$

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Hence,

$$1 + (N-1)(1-\varepsilon^2) \le \mathcal{R}^+_{\Phi_3} < N.$$

It is not all Good News

We can also compute

$$\mathcal{R}_{\Phi_3}(e_2) = \sum_{i=1}^{N} \|P_{\langle\varphi_i\rangle}(e_2)\|^2 = \sum_{i=2}^{N} |\langle e_2, \sqrt{1-\varepsilon^2}e_1 + \varepsilon e_i\rangle|^2 = \varepsilon^2,$$

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Hence,

$$0 < \mathcal{R}_{\Phi_3}^{-} \leq \varepsilon^2.$$

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Our results show that this frame can be partitioned into $N-1$ linearly independent sets. Again, we see that we can do better than this by merely taking the whole frame which happens to be linearly independent.

However, our redundancy notion is giving pretty good information even in this extreme case. i.e. The very fact that

$$0 < \mathcal{R}^-_{\Phi_3} \leq \varepsilon^2,$$

tells us that this frame is heavily piled up on itself.