An Introduction to Filterbank Frames

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Overview

This goal of this talk is to provide a mathematical overview of filterbank frame theory, much of which stems from literature in engineering. The talk will begin by considering filterbanks with integer sampling and then move on to consider work with rational sampling factors.

- Preliminaries
- Introduction
- The Polyphase Representation
- Filterbank Frame Theory
- Filterbanks with Rational Sampling
- Further Exploration

The Concept of a Frame

Definition

A *frame* for a separable Hilbert space $\mathbb{H}$ is a collection $\{e_j\}_{j \in \mathbb{Z}}$ for which there exist constants $0 < A \leq B < \infty$ such that

$$A\|x\|^2 \leq \sum_{j \in \mathbb{Z}} |\langle x, e_j \rangle|^2 \leq B\|x\|^2 \quad \text{for all } x \in \mathbb{H}.$$

- Any collection for which the right-hand inequality holds for some $B < \infty$ is called a *Bessel system*.
- If it is possible to choose $A = B$ then the frame is said to be *tight*.
- For any Bessel system the *frame operator* is defined by

$$Sx = \sum_{j \in \mathbb{Z}} \langle x, e_j \rangle e_j.$$

Frame Inversion

The *frame coefficients* $\{\langle x, e_j\rangle\}_j$ uniquely determine $x \in \mathbb{H}$. One may recover x from the frame coefficients as follows.

- Tight frame: $S = AI_{\mathbb{H}}$, so $x = A^{-1}Sx$.
- General case:

Theorem (Frame Algorithm)

Given a frame $\{e_j\}_j$ one may recover x from its frame coefficients as follows. Define $x_0 = 0$ and

$$x_n = x_{n-1} + \frac{2}{A+B}S(x - x_{n-1}).$$

Then, $\|x - x_n\| \leq \left(\frac{B-A}{B+A}\right)^n \|x\|$.

Note: Sx is completely determined by the frame coefficients.

Frame Inversion

One may also construct a *dual frame*, $\{\tilde{e}_j\}_j$ such that

$$x = \sum_{j\in\mathbb{Z}} \langle x, e_j\rangle \tilde{e}_j \quad \text{for all } x \in \mathbb{H}.$$

In fact, notice that one may choose $\tilde{e}_j = S^{-1}e_j$, since

$$\sum_{j\in\mathbb{Z}} \langle x, e_j\rangle S^{-1}e_j = S^{-1}Sx = x.$$

Moreover, $S\tilde{e}_j = e_j$, allowing one to iteratively approximate $\tilde{e}_j$ by modifying the frame algorithm:

$$x_n = x_{n-1} + \frac{2}{A+B}S(\tilde{e}_j - x_{n-1}) \quad \text{becomes} \quad x_n = x_{n-1} + \frac{2}{A+B}(e_j - Sx_{n-1})$$

and, now, $x_n \to \tilde{e}_j$ as $n \to \infty$.

The Hilbert Space - $\ell^2(\mathbb{Z})$

- Filterbanks act on sequences, so the Hilbert space in question is $\ell^2(\mathbb{Z})$.
- Fourier transform:

$$\hat{x}(\xi) = \sum_{n\in\mathbb{Z}} x(n)e^{-2\pi i n\xi}, \xi \in \mathbb{T}.$$

- Convolution:

$$(x * y)(n) = \sum_{k\in\mathbb{Z}} x(k)y(n-k) \qquad \widehat{x*y}(\xi) = \hat{x}(\xi)\,\hat{y}(\xi).$$

- An *ideal filter* is y such that $\hat{y} = \chi_E$ some $E \subseteq \mathbb{T}$. Convolution of x with y isolates the content of $\hat{x}$ inside E. (filters out the rest)
- Denote by $\tilde{x}$ the sequence such that $\tilde{x}(n) = x(-n)$.

Filterbanks Frames - Ingredients

- Filters: $h_0, h_1, \ldots, h_{M-1} \in \ell^2(\mathbb{Z})$
  Note: Filters with finite support are often desired.
- Candidate frame:

$$F = \{\tilde{h}_j(\cdot - nN) : 0 \leq j \leq M-1, n \in \mathbb{Z}\}.$$

- Interpretation via convolution:

$$(x * \overline{h}_j)(nN) = \sum_{k\in\mathbb{Z}} x(k)\, \overline{h_j(nN-k)} = \sum_{k\in\mathbb{Z}} x(k)\, \overline{\tilde{h}_j(k-nN)} = \langle x, \tilde{h}_j(\cdot - nN)\rangle$$

The frame coefficients are given by samples of the convolutions with $\overline{h}_j$.

Filterbanks Frames - Sampling

- Downsampling & Upsampling

$$(\downarrow_N x)(n) = x(nN) \qquad (\uparrow_N x)(n) = \begin{cases} x(m) & n = mN \\ 0 & \text{otherwise} \end{cases}$$

The two operators are adjoint.

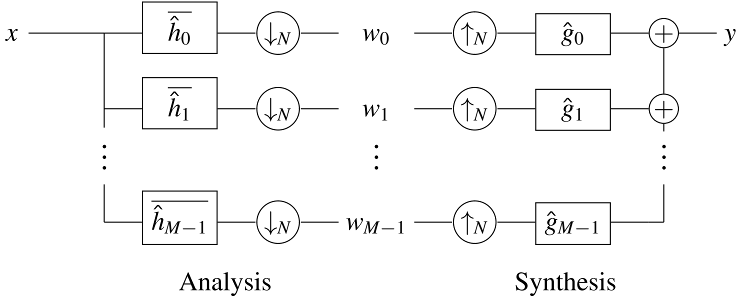
- The frame coefficients are captured by the coordinates of

$$w_j = \downarrow_N (x * \overline{h}_j), \quad 0 \le j \le M-1,$$

so that

$$\sum_{j=0}^{M-1} \sum_{n\in\mathbb{Z}} \left| \left\langle x, \tilde{h}_j(\cdot - nN) \right\rangle \right|^2 = \sum_{j=0}^{M-1} \|w_j\|^2$$

Filterbanks Frames - Schematic



- When $M = N$ the filterbank is said to be *critically sampled*. This case seems to have garnered the greatest amount of attention.
- When $M > N$ the filterbank is said to be *oversampled*, as, in some sense, the analysis retains more samples than are necessary.

Some Remarks

- If $g_j = h_j$, $0 \leq j \leq M-1$, then the synthesis stage is dual to the analysis stage and y is merely Sx. (S being the frame operator.)
- Typically, h_0 is a *low-pass filter*, i.e., $\hat{h}(\xi)$ is continuous and nonzero near $\xi = 0$. The remaining filters will generally possess a zero at $\xi = 0$, leading to many small coefficients when the original signal is smooth.
- The signal w_0 is a smoothed version of the input signal and is often further analyzed using the same filterbank. (However, wavelet packets allow analysis of any of the components w_j, $0 \leq j \leq M-1$.)
- Engineering literature [5] often focuses on *perfect reconstruction*, where $y = x$. This corresponds to the class of Parseval frames (tight frames with $A = B = 1$) or the class of dual frames when separate filters are used in analysis and synthesis.

Fourier Identities

Standard arguments lead to the following identities.

- Downsampling:

$$\widehat{\downarrow_N x}(\xi) = \frac{1}{N} \sum_{k=0}^{N-1} \hat{x}\left(\frac{\xi}{N} + \frac{k}{N}\right)$$

- Upsampling:

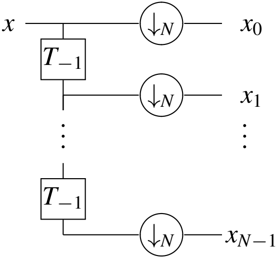
$$\widehat{\uparrow_N x}(\xi) = \hat{x}(N\xi)$$

- Translation: $(T_k x)(n) = x(n-k)$

$$\widehat{T_k x}(\xi) = e^{-2\pi i m \xi}\hat{x}(\xi)$$

Forward Polyphase Transform

$$\begin{aligned}\hat{x}(\xi) &= \sum_{k=0}^{N-1}\sum_{n\in\mathbb{Z}} x(nN+k)e^{-2\pi i(nN+k)\xi}\\ &= \sum_{k=0}^{N-1} e^{-2\pi ik\xi}\sum_{n\in\mathbb{Z}} x(nN+k)e^{-2\pi inN\xi}\\ &= \sum_{k=0}^{N-1} e^{-2\pi ik\xi}\hat{x}_k(N\xi),\end{aligned}$$



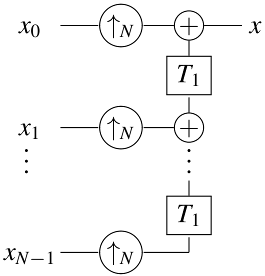
where

$$\hat{x}_k(\xi) = \sum_{n\in\mathbb{Z}} x(nN+k)e^{-2\pi in\xi}.$$

The sequences x_k, $0 \le k \le N-1$, are the *polyphase components* of x.

Inverse Polyphase Transform

The Polyphase Transform is unitary, hence inversion comes via the adjoint.



It is convenient to analyze filterbanks using polyphase components of both the signal and the filters.

Polyphase Matrix

The *polyphase matrix* for a filterbank is given by

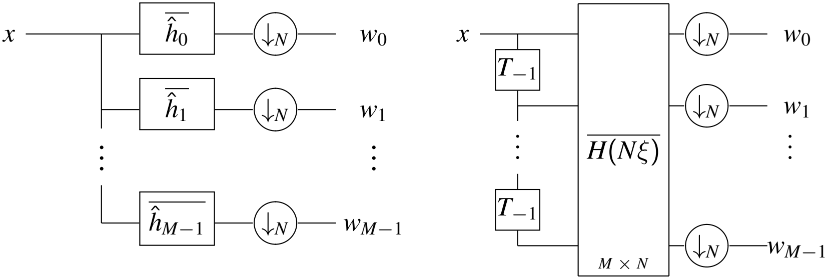
$$H(\xi) = \begin{pmatrix} \hat{h}_{0,0}(\xi) & \cdots & \hat{h}_{0,N-1}(\xi) \\ \vdots & \ddots & \vdots \\ \hat{h}_{M-1,0}(\xi) & \cdots & \hat{h}_{M-1,N-1}(\xi) \end{pmatrix},$$

where $h_{j,k}$ is the kth polyphase component of filter h_j.

- Polyphase components are particularly convenient when the filters are finitely supported.
- If separate filters are used for analysis and synthesis one works with corresponding analysis and synthesis polyphase matrices.

Polyphase Representation of Analysis

The following diagrams are equivalent.



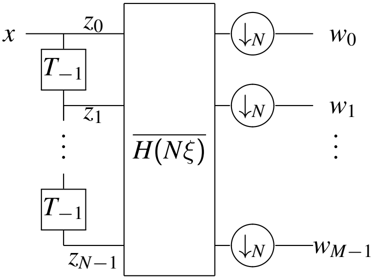
Observe

$$\hat{x}(\xi)\overline{\hat{h}_j(\xi)} = \hat{x}(\xi)\sum_{k=0}^{N-1}\overline{\hat{h}_{j,k}(N\xi)e^{-2\pi ik\xi}} = \sum_{k=0}^{N-1}\overline{\hat{h}_{j,k}(N\xi)}\,\widehat{T_{-k}x}(\xi)$$

which is the jth row of $\overline{H(N\xi)}$ multiplied by the column vector of modulated versions of $\hat{x}$.

Noble Identity (Analysis)

The following diagrams are equivalent.

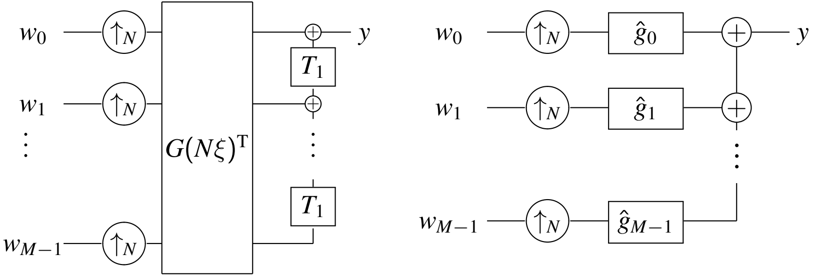


This is justified by the following calculation.

$$\hat{w}_j(\xi) = \frac{1}{N}\sum_{\ell=0}^{N-1}\sum_{k=0}^{N-1}\overline{\hat{h}_{j,k}(N(\frac{\xi}{N}+\frac{\ell}{N})}\hat{z}_k(\frac{\xi}{N}+\frac{\ell}{N}) = \sum_{k=0}^{N-1}\overline{\hat{h}_{j,k}(\xi)}\frac{1}{N}\sum_{\ell=0}^{N-1}\hat{z}_k(\frac{\xi}{N}+\frac{\ell}{N})$$

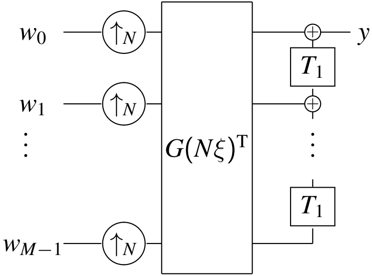
Polyphase Representation of Synthesis

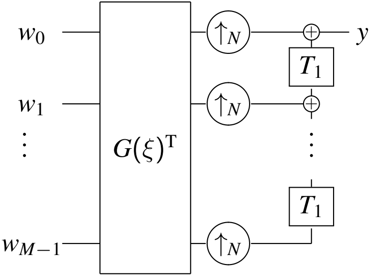
Similar calculations show that the following diagrams are equivalent.



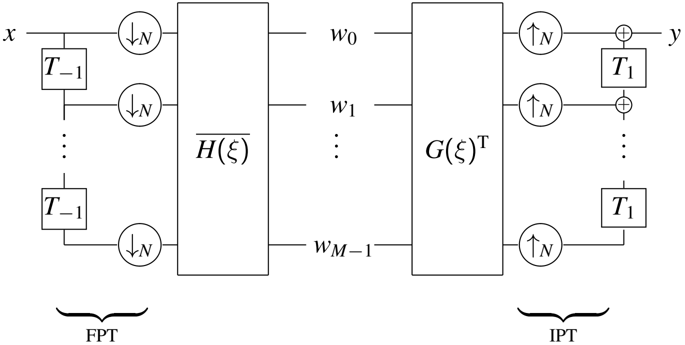
Noble Identity (Synthesis)

Finally, the following two diagrams are also equivalent.





Filterbank - Polyphase Reprsentation



Recall that the polyphase transform is a unitary transformation. Hence, the frame properties of the filterbank are completely determined by the polyphase matrix.

Perfect Reconstruction

Cvetković and Vetterli made one of the early examinations of filterbank frame properties using the polyphase representation [2].

Theorem

A filterbank has the perfect reconstruction property if and only if

$$G(\xi)^* H(\xi) = I_N, \quad a.e.\ \xi.$$

Proposition

Assume $G = H$. The filterbank implements a tight frame expansion if and only if $H(\xi)$ is paraunitary, i.e.,

$$H(\xi)^* H(\xi) = cI_N, \quad a.e.\ \xi.$$

Frame Properties

Assume $G = H$.

Proposition

If the filters have finite support, then the corresponding filterbank implements a frame expansion if and only if its polyphase matrix H is of full rank on $\mathbb{T}$.

Theorem

The frame bounds for the filterbank are described by

$$A = \operatorname*{ess\,inf}_{\xi \in \mathbb{T}} \lambda(\xi) \quad \text{and} \quad B = \operatorname*{ess\,sup}_{\xi \in \mathbb{T}} \Lambda(\xi),$$

where $\lambda(\xi)$ and $\Lambda(\xi)$, respectively, are the minimum and maximum eigenvalues of $H^*(\xi)H(\xi)$.

Why Rational Sampling?

- Integer sampling: Critically sampled filterbanks seek to decompose a signal into components representing frequency bands.
- Sampling by 2: Ideal filters

$$\hat{h}(\xi) = \chi_{[-\frac{1}{4},\frac{1}{4}]}(\xi) \qquad \hat{g}(\xi) = \chi_{[-\frac{1}{2},-\frac{1}{4}]\cup[\frac{1}{4},\frac{1}{2}]}(\xi)$$

  Frequency spectrum is divided into two equal bands, the low-pass and high-pass bands.
- Sampling by larger integer factors results in smaller subbands, still with one low-pass band and, now, multiple high-pass bands.
- The goal of rational sampling is to achieve unequal widths for subbands and to permit subbands of width greater than $\frac{1}{2}$.

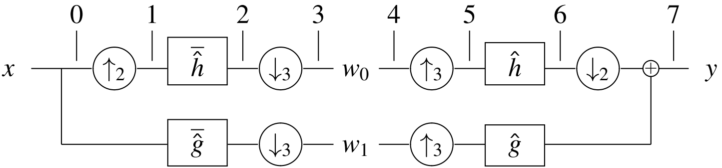
Filterbanks with Rational Sampling

Kovačević and Vetterli [4]:

- Motivated by applications where analysis via unequal subbands could be useful, e.g., analysis of speech or music.
- Given rational numbers $r_0, \ldots, r_{M-1}$ whose sum is one, the goal of the rational filterbank is to decompose into subbands of width r_j, $0 \leq j \leq M-1$. The first channel will be low-pass.
- Rational sampling is accomplished by combining downsampling and upsampling operations with different integer rates.
- Downsampling and upsampling operations commute when the rates are relatively prime. This leads to many equivalent filterbanks.

Perfect Reconstruction - An Example

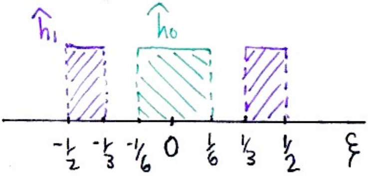
Consider the following $(\frac{2}{3}, \frac{1}{3})$ filterbank.



- Ideal filters of Kovačević and Vetterli:

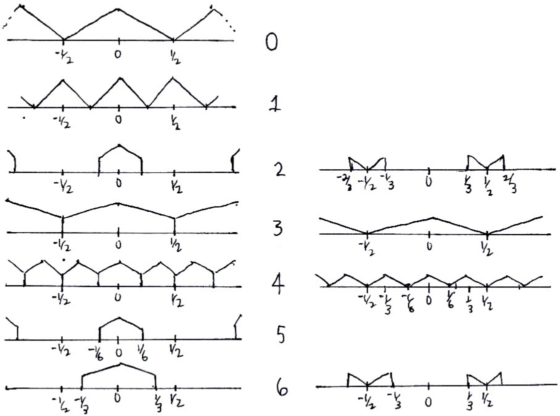
$$\hat{h}(\xi) = \chi_{[-\frac{1}{6},\frac{1}{6}]}(\xi) \qquad \hat{g}(\xi) = \chi_{[-\frac{1}{2},-\frac{1}{3}]\cup[\frac{1}{3},\frac{1}{2}]}(\xi)$$

- Pictorially: $(h_0 = h \;\&\; h_1 = g)$



Perfect Reconstruction - Step by Step

Perfect reconstruction will be justified through sketches.



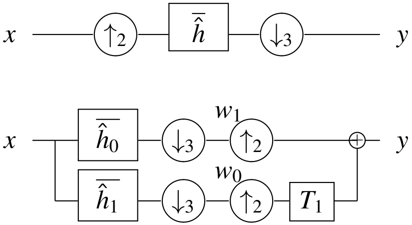
Adding the two signals after stage 6 recovers the original signal.

A Few Remarks

Further remarks on the work of Kovačević and Vetterli:

- Direct methods (as above) are not successful in all cases. While the $(\frac{2}{3}, \frac{1}{3})$ case is solvable using ideal filters, the $(\frac{1}{3}, \frac{2}{3})$ case is not.
- Indirect methods were used which involved the study of equivalent filterbanks.
  - The "analysis" stage can be expressed as a combination of integer-sampled analysis and synthesis components, e.g., 3-channel analysis followed by 2-channel synthesis.
  - Existing perfect reconstruction filters could then be used which have finite support.
- Only perfect recontruction filterbanks were considered.

Equivalent Low-Pass Channels



Bayram and Selesnick [1] noticed that a modified polyphase representation is convenient:

$$h_0(n) = h(2n) \qquad h_1(n) = h(2n+3)$$

It is sufficient to show that $w_0(n) = y(2n)$ and $w_1(n) = y(2n+1)$.

Equivalent Low-Pass Channels (proof)

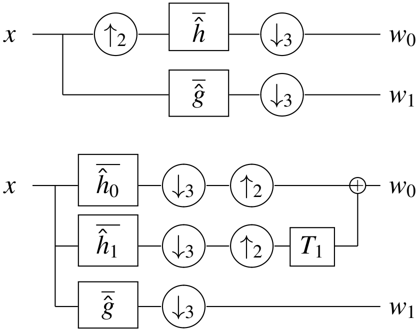
$$
\begin{aligned}
y(n) &= (\uparrow_2 x * h)(3n) \\
&= \sum_{m\in\mathbb{Z}} (\uparrow_2 x)(m) h(3n-m) \\
&= \sum_{m\in\mathbb{Z}} x(m) h(3n-2m)
\end{aligned}
$$

$$
\begin{aligned}
w_0(n) &= (x * h_0)(3n) \\
&= \sum_{m\in\mathbb{Z}} x(m) h_0(3n-m) \\
&= \sum_{m\in\mathbb{Z}} x(m) h(6n-2m) \\
&= y(2n)
\end{aligned}
$$

$$
\begin{aligned}
w_1(n) &= (x * h_1)(3n) \\
&= \sum_{m\in\mathbb{Z}} x(m) h_1(3n-m) \\
&= \sum_{m\in\mathbb{Z}} x(m) h(6n-2m+3) \\
&= \sum_{m\in\mathbb{Z}} x(m) h(3(2n+1)-2m) \\
&= y(2n+1)
\end{aligned}
$$

Equivalent Filterbanks

This shows that the following filterbanks are equivalent.



Observations

- The merging of the two low-pass components is a unitary transformation and, as such, does not affect frame properties.
- The frame properties are determined by the standard 3-band polyphase matrix,

$$H(\xi) = \begin{pmatrix} \hat{h}_{0,0}(\xi) & \hat{h}_{0,1}(\xi) & \hat{h}_{0,2}(\xi) \\ \hat{h}_{1,0}(\xi) & \hat{h}_{1,1}(\xi) & \hat{h}_{1,2}(\xi) \\ \hat{g}_0(\xi) & \hat{g}_1(\xi) & \hat{g}_2(\xi) \end{pmatrix}.$$

  Thus, any paraunitary H will lead to perfect reconstruction.
- Bayram and Selesnick have used this approach to construct critically sampled perfect reconstruction filterbanks, although the low-pass filters become somewhat rough upon iteration [1].
- Bayram and Selesnick [1] then showed that oversampling leads to perfect reconstruction filterbanks which behave better under multiple iterations.

Areas for Further Exploration

Connecting filterbanks and function systems:

Daubechies [3] discussed the relationship between orthonormal wavelet systems using dilation $\frac{3}{2}$ and $(\frac{2}{3}, \frac{1}{3})$ perfect reconstruction filterbanks. Such wavelets cannot have compact support, yet filterbanks with finite support do exist. Can such filterbanks be associated with some kind of function system?

Along the same lines, the oversampled rational filterbanks of Bayram and Selesnick may lead to compactly supported frame wavelets for $L^2(\mathbb{R})$ with rational dilations.

Areas for Further Exploration

Understanding iterated filterbank frames:

Bayram and Selesnick restrict attention to filterbanks with perfect reconstruction. In the critically sampled case, is it possible to obtain filters which perform well under multiple iterations by considering non-tight filterbank frames?

In general, how does one evaluate the performance of a generic filterbank frame under multiple iterations? (Perfect reconstruction makes this a trivial concern.)

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