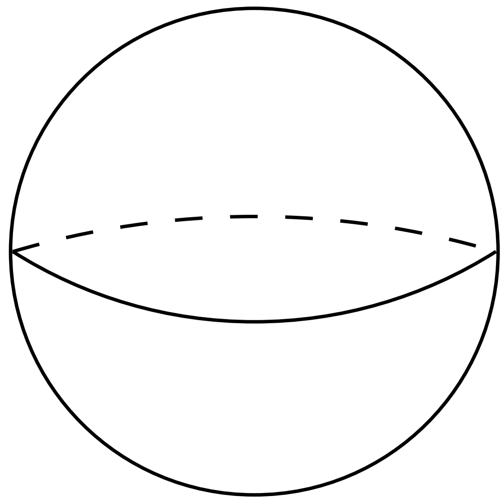
The Nonholonomy of the Rolling Sphere



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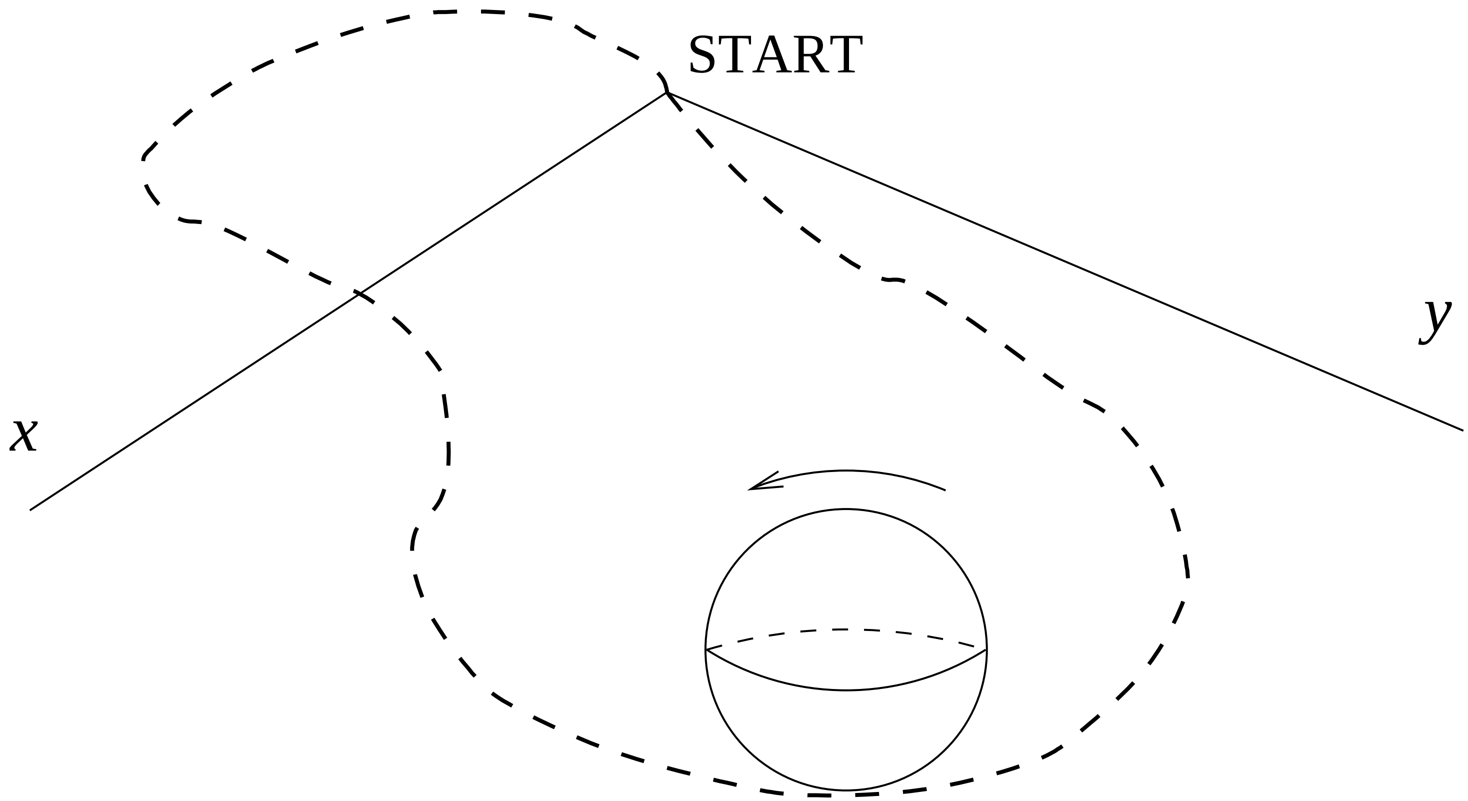
Acknowledgement:

This talk is based on results presented by Professor Robert Bryant of Duke University during a lecture entitled “One Hundred Years of Holonomy” in October of 2003 as part of *150 Years of Progress in the Mathematical Sciences, A Conference to Celebrate the Sesquicentennial of Washington University*.

This version of the talk has benefited from the input of the referees of [2], the editor of the MAA Monthly, as well as the summer 2006 REU students in Mathematics at Cornell University.

Problem Statement: Given a pair of orientations for a sphere resting on a plane, is there a closed path along which one can roll the sphere (without slipping or twisting), starting with the first orientation, and return to the origin with the sphere in the second orientation?



Origin of the Problem [1]:

- In the late nineteenth century the notion of holonomy was introduced to describe certain features of mechanical systems. A mechanical system is said to be *non-holonomic* with respect to a given constrained motion if the system can move between any two states without violating the constraint. Otherwise, the system is said to be *holonomic* with respect to the constraint.

- In our case, the mechanical system is the sphere on the plane, the possible states include all possible orientations of the sphere at a fixed point on the plane, and the constraint on the motion is that the sphere must be rolled (with no slipping or twisting) about a closed path. (a closed path begins and ends in the same place)

A Simpler Example [1]:

- System: A particle in the xy-plane.
- States: The position of the particle may be any coordinate (x, y).
- Constraint: The velocity vector of the particle must be perpendicular to its position vector.

If the position vector is $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$ and $\vec{v}(t) = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j}$ then this system must obey the identity:

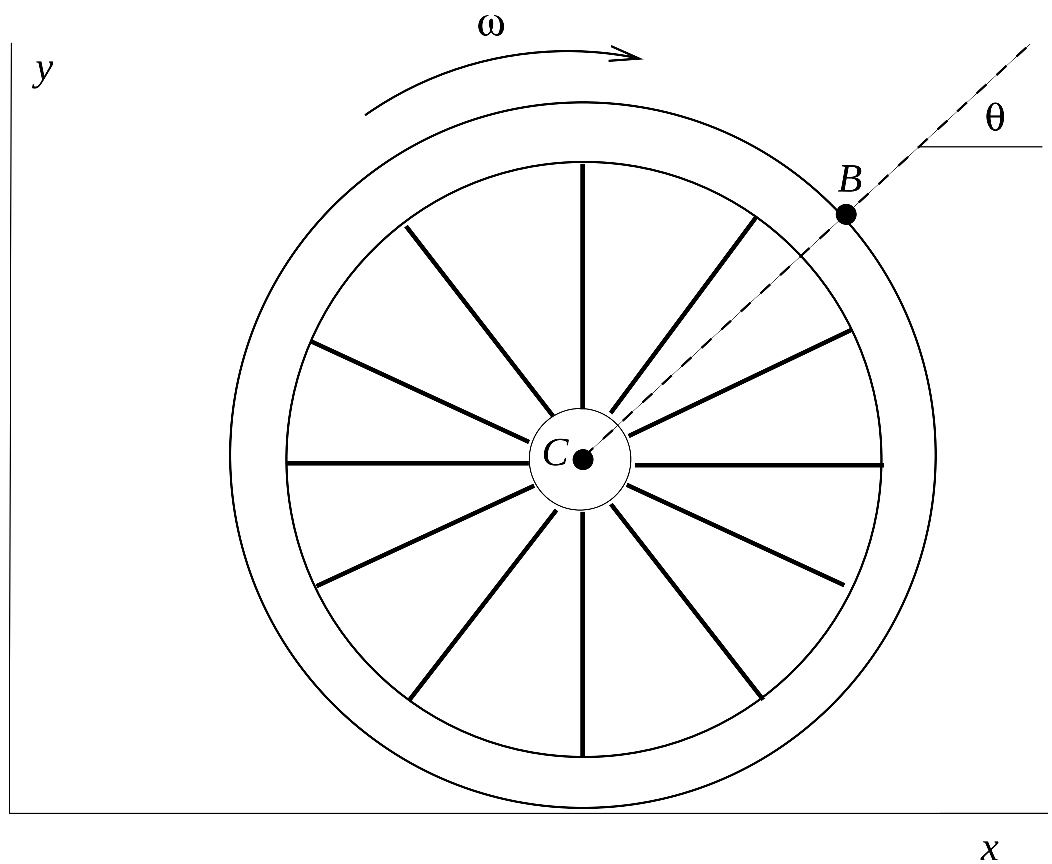
$$x\frac{dx}{dt} + y\frac{dy}{dt} = 0,$$

which means $y\frac{dy}{dx} = -x$. This is a separable differential equation whose general solution is $x^2 + y^2 = C$, for any non-negative constant C.

Conclusion: The particle is limited to circular motion and, hence, the system is holonomic.

A Rolling Wheel:

Consider a wheel of radius R rolling with angular frequency ω_0.



Still a Rolling Wheel:

- The velocity of the center C will be

$$\vec{v}_C = \omega_0 R\vec{i} + R\vec{j}.$$

- The velocity at an arbitrary point on the outside of the wheel, B, will be given by

$$\vec{v}_B = \vec{v}_C + \vec{v}_{B/C} = \vec{v}_C + \omega_0\vec{k} \times \vec{r}_{B/C}.$$

- Notice that if $\vec{r}_{B/C} = R\vec{j}$, then $\vec{v}_B = 2\omega_0 R\vec{i}$ while if $\vec{r}_{B/C} = -R\vec{j}$, then $\vec{v}_B = \vec{0}$.

- This last observation characterizes rolling without slipping, i.e., that **the point of the wheel in contact with the ground must have zero instantaneous velocity**.

The Rolling Sphere: Equations of Motion (1 of 4)

- Let $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j}$ be the path in the plane about which the sphere is to be rolled. (The sphere's center, C, will follow $\vec{r}_C(t) = \vec{r}(t) + \vec{k}$.)

- The first problem is to determine what rotation of the sphere, $\vec{\omega}(t)$, produces this motion under the constraint of no slipping and no twisting.

- No slipping means the velocity at the bottom of the sphere is zero:

$$\begin{aligned}\vec{0} = \vec{v}_B(t) &= \vec{r}'_C(t) + \vec{\omega}(t) \times \vec{r}_{B/C} \\ &= \vec{r}'_C(t) + \vec{\omega}(t) \times (-\vec{k}) \\ &= f'(t)\vec{i} + g'(t)\vec{j} + \omega_x(t)\vec{j} - \omega_y(t)\vec{i}.\end{aligned}$$

- No twisting means $\vec{\omega}(t) \cdot \vec{k} = 0$.

The Rolling Sphere: Equations of Motion (2 of 4)

- The two previous observations now imply that

$$\vec{\omega}(t) = -g'(t)\vec{i} + f'(t)\vec{j}.$$

- Since the path $\vec{r}(t)$ is closed (starts and ends at the origin for our purposes) in order to determine the final orientation of the sphere we only require information about the relative motion of points on the sphere with respect to the center.

- In other words, given the rotation $\vec{\omega}(t)$ we can find the final state by considering the sphere as if it were spinning in place.

- The next task is to characterize the motion of an arbitrary point on the sphere under such a rotation.

The Rolling Sphere: Equations of Motion (3 of 4)

- The motion of an arbitrary point P will be given by

$$\vec{v}_P(t) = \vec{\omega}(t) \times \vec{r}_{P/C}(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -g'(t) & f'(t) & 0 \\ x_P(t) & y_P(t) & z_P(t) \end{vmatrix}.$$

- This leads to the following system of differential equations:

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{pmatrix} = \begin{pmatrix} f'(t)z(t) \\ g'(t)z(t) \\ -f'(t)x(t) - g'(t)y(t) \end{pmatrix}.$$

The Rolling Sphere: Equations of Motion (4 of 4)

- Recall that the goal is to solve this system of equations given for a given path given an initial condition (x_0, y_0, z_0).
- For example, consider a linear path $\vec{r}(t) = at\vec{\imath} + bt\vec{\jmath}$, where $a, b \in \mathbb{R}$. In this case, the equations of motion are

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{pmatrix} = \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & a \\ -a & -b & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}, \tag{1}$$

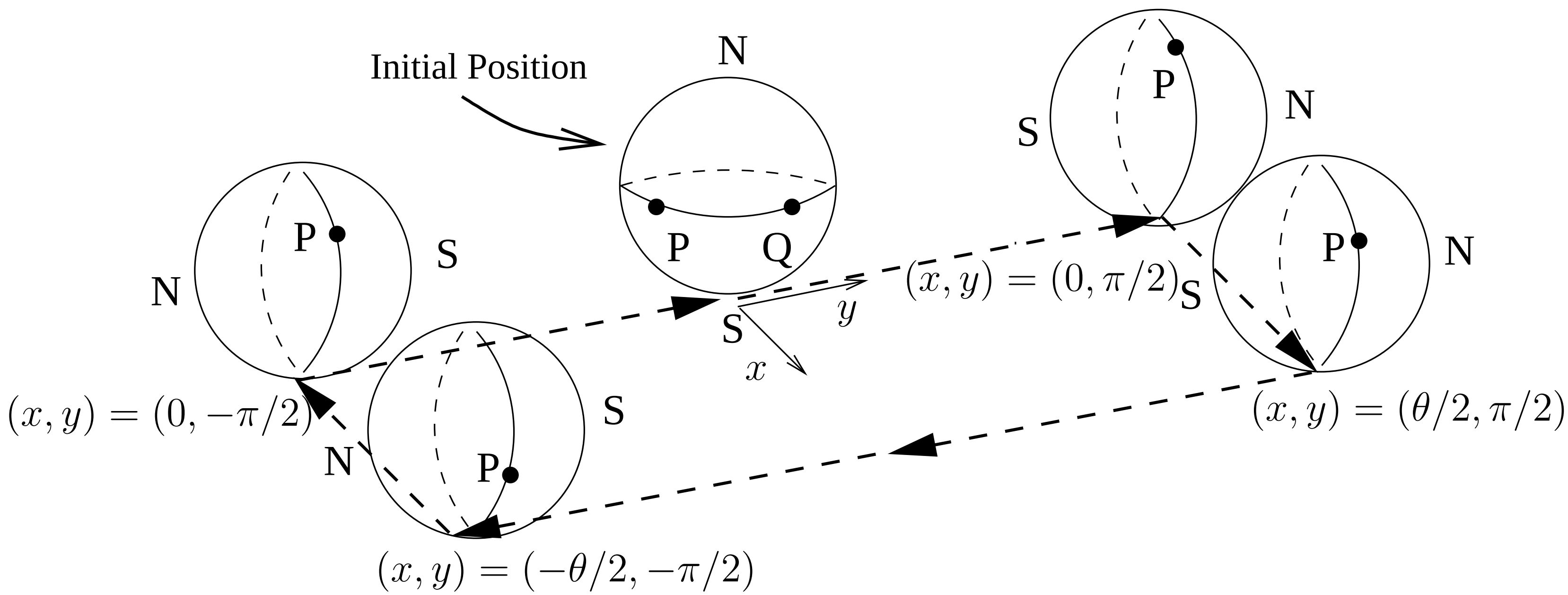
  which are easily solved due to the constant coefficient matrix.
- Outside the realm of piecewise linear paths, however, solution of the equations of motion will be more challenging.

The Holonomy Question:

- Assume the sphere begins at the origin.
- An orientation of the sphere is determined by two features:
  - the point on the sphere which occupies the North pole;
  - the location of a point on the equator (i.e., an angle about the z-axis).
- One could prove that the system is nonholonomic by:
  - showing that one can achieve any desired rotation about the z-axis without disturbing the North Pole by rolling through a closed path;
  - showing that any point on the sphere can be relocated to the North Pole by rolling through a closed path.

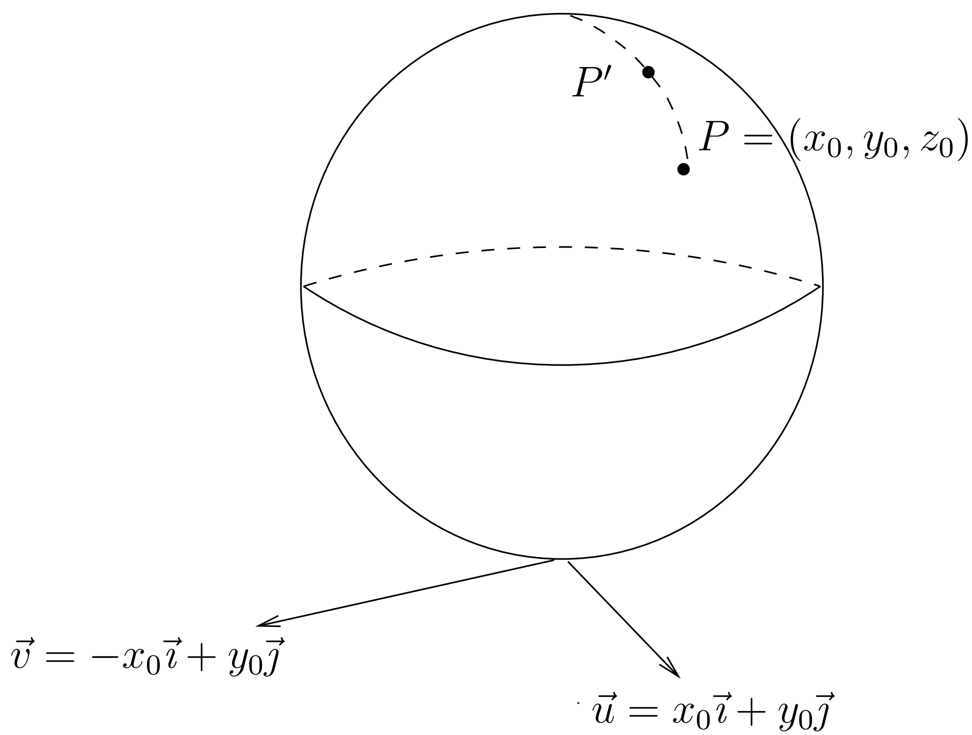
Lemma 1: Rotation about the z-axis:

The no-twisting constraint prevents us from directly rotating the sphere about the z-axis, but we can accomplish an equivalent transformation by rolling the sphere through the path depicted below.



Lemma 2: Relocation to the North Pole (1 of 2)

A given point P on the sphere can be relocated to the North Pole by rolling the sphere through a rectangular path similar to that of Lemma 1.



Lemma 2: Relocation to the North Pole (2 of 2)

Description of the path:

1. Roll the sphere in the $-\vec{u}$ direction (along a great circle) until the midpoint P' of the connecting arc is at the North Pole;
2. Roll the sphere in the $\vec{v}$ direction π units, relocating P' to the South Pole;
3. Roll the sphere in the $\vec{u}$ direction so that P is at the South Pole;
4. Roll the sphere in the $-\vec{v}$ direction by π units, bringing the sphere back to the origin and rotating P to the North Pole.

Nonholonomy of the Rolling Sphere

Combining Lemmas 1 and 2 we have shown that any given orientation of the sphere is achievable at a fixed point on the plane by way of rolling with no slipping or twisting. (More details can be found in [2].)

Theorem 1. *The sphere resting on the plane constrained to rolling through closed paths with no slipping or twisting is a nonholonomic system.*

Thus far the equations of motion have been put to little use. What else can we do with them?

A Circular Path for the Rolling Sphere:

- Suppose the sphere is rolled through a circle of radius R which begins and ends at the origin:

$$\vec{r}(t) = R(1 - \cos t)\vec{i} + R \sin t \vec{j}, \quad 0 \leq t \leq 2\pi.$$

- The system of differential equations becomes:

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{pmatrix} = \begin{pmatrix} R \sin t\, z(t) \\ R \cos t\, z(t) \\ -R \sin t\, x(t) - R \cos t\, y(t) \end{pmatrix}.$$

This system looks harder than it really is and can be simplified by using a rotating coordinate system.

Rotating Coordinates: (1 of 3)

- Introduce a rotating coordinate system (x_r, y_r) via:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cos\left(\frac{\pi}{2} - t\right) & -\sin\left(\frac{\pi}{2} - t\right) \\ \sin\left(\frac{\pi}{2} - t\right) & \cos\left(\frac{\pi}{2} - t\right) \end{pmatrix} \begin{pmatrix} x_r(t) \\ y_r(t) \end{pmatrix}.$$

  Also, let $z_r(t) = z(t)$.

- The (x_r, y_r, z_r) coordinates are time dependent and rotate with the sphere through the circular path. Under this change of variables the equations of motion become:

$$\begin{pmatrix} \frac{dx_r}{dt} \\ \frac{dy_r}{dt} \\ \frac{dz_r}{dt} \end{pmatrix} = \begin{pmatrix} -y_r(t) + Rz_r(t) \\ x_r(t) \\ -Rx_r(t) \end{pmatrix} = \begin{pmatrix} 0 & -1 & R \\ 1 & 0 & 0 \\ -R & 0 & 0 \end{pmatrix} \begin{pmatrix} x_r(t) \\ y_r(t) \\ z_r(t) \end{pmatrix}.$$

Rotating Coordinates: (2 of 3)

- The equations of motion are now given by a first order, linear system of differential equations, whose general solution is:

$$\vec{x}_r(t) = C_1\vec{x}_1(t) + C_2\vec{x}_2(t) + C_3\vec{x}_3(t),$$

$$\vec{x}_1(t) = \frac{1}{\sqrt{R^2+1}}\begin{pmatrix} 0 \\ R \\ 1 \end{pmatrix}$$

$$\vec{x}_2(t) = \frac{1}{\sqrt{R^2+1}}\begin{pmatrix} 0 \\ 1 \\ -R \end{pmatrix}\cos(\sqrt{R^2+1}t) - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\sin(\sqrt{R^2+1}t)$$

$$\vec{x}_3(t) = \frac{1}{\sqrt{R^2+1}}\begin{pmatrix} 0 \\ 1 \\ -R \end{pmatrix}\sin(\sqrt{R^2+1}t) + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\cos(\sqrt{R^2+1}t).$$

Rotating Coordinates: (3 of 3)

- The coefficients for the particular solution are given by:

$$\begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} 0 & \frac{R}{\sqrt{R^2+1}} & \frac{1}{\sqrt{R^2+1}} \\ 0 & \frac{1}{\sqrt{R^2+1}} & -\frac{R}{\sqrt{R^2+1}} \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_r(0) \\ y_r(0) \\ z_r(0) \end{pmatrix}.$$

- If $(x_r(0), y_r(0), z_r(0)) = (0, 0, 1)$ (the North Pole) then the particular solution is given by

$$\vec{x}_r(t) = \frac{1}{R^2+1} \left[\begin{pmatrix} 0 \\ R \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ R \\ -R^2 \end{pmatrix} \cos(\omega t) + \sqrt{R^2+1} \begin{pmatrix} R \\ 0 \\ 0 \end{pmatrix} \sin(\omega t) \right],$$

where $\omega = \sqrt{R^2 + 1}$.

A Circular Alternative to Lemma 2:

Proposition 2. *For any given point P on the sphere there exists a piecewise differentiable path $\vec{r}(t)$ $(0 \le t \le T < \infty)$, composed of circular components, such that rolling the sphere through the path $\vec{r}(t)$ relocates P to the North Pole.*

The proof will make use of the axial symmetry of the sphere. Let $\vec{u}$ and $\vec{v}$ be the images, respectively, of $\vec{\imath}$ and $\vec{\jmath}$ under a rotation by θ in the xy-plane. Replace the standard circular path, above, by

$$\vec{r}(t) = R(1 - \cos t)\vec{u} + R \sin t \vec{v} \quad (0 \le t \le 2\pi). \tag{2}$$

Suppose that rolling the sphere through the original circle moves a point P on the sphere to another point, Q. If $\tilde{P}$ and $\tilde{Q}$ are the images of P and Q, respectively, under the same rotation used above, then rolling the sphere through the path (2) will move $\tilde{P}$ to $\tilde{Q}$.

Proof of Proposition 2:

- Fix $P = (x_0, y_0, z_0)$ in the closed northern hemisphere ($z_0 \geq 0$).
- The circular path relocates the North Pole to the altitude

$$z(2\pi) = \left[1 + R^2 \cos\left(2\pi\sqrt{R^2+1}\right)\right] / (R^2+1). \qquad (3)$$

- The final altitude varies continuously with R:

$$R = 0 \mapsto z(2\pi) = 1 \quad \text{and} \quad R = \frac{\sqrt{5}}{2} \mapsto z(2\pi) = -\frac{1}{9}.$$

  The Intermediate Value Theorem implies that any desired nonnegative altitude can be achieved.

- For every nonnegative altitude, there exists a point P with that altitude which can be moved to the North Pole by a circular path.

Proof of Proposition 2:

- Given another point, $\tilde{P}$, with the same altitude as P it is possible to rotate the circular path so that $\tilde{P}$ is relocated to the North Pole instead of P. Thus, any point in the northern hemisphere can be relocated to the North Pole by a single circular path.
- Now fix $P = (x_0, y_0, z_0)$ in the southern hemisphere ($z_0 < 0$). Two circular paths will be used in sequence to move P to the North Pole.
- Consider the great circle containing P and the North Pole. Observe that the shorter arc between P and the North Pole has length less than or equal to π.
- The midpoint of this arc lies in the closed northern hemisphere, so there exists a circular path which relocates the midpoint of the arc to the North Pole. It follows that this path also moves P to closed northern hemisphere. $\square$

Necessity of Two Circular Paths:

Proposition 3. *The North and South Poles cannot be exchanged by rolling the sphere (with no slipping or twisting) through a single circular path.*

Proof. As in the proof of Proposition 2, the main tool is the general solution for the circular path obtained above. For a fixed radius $R > 0$ the sphere returns to its starting point when $t = 2\pi$ and the image of the North Pole satisfies

$$-1 + \frac{2}{R^2 + 1} = \frac{1 - R^2}{R^2 + 1} \leq z(2\pi) \leq \frac{1 + R^2}{R^2 + 1} = 1.$$

Thus, $z(2\pi)$ can never equal -1, showing that the North and South Poles cannot be interchanged via rolling along a single circular path.

□

Conclusion:

For more information about this subject or the details of the preceding arguments one could look at the following references.

References

[1] Robert Bryant. *One Hundred Years of Holonomy*. Washington University in St. Louis, October 2003.

[2] Brody Dylan Johnson. The nonholonomy of the rolling sphere. *Amer. Math. Monthly*, 114(6):500–508, 2007.