Chapter 5

Consequences of Cauchy’s Theorem

If things are nice there is probably a good reason why they are nice: and if you do not know at least one reason for this good fortune, then you still have work to do.
Richard Askey

5.1 Extensions of Cauchy’s Formula

We now derive formulas for $f'$ and $f''$ which resemble Cauchy’s formula (Theorem 4.10).

**Theorem 5.1.** Suppose $f$ is holomorphic on the region $G$, $w \in G$, and $\gamma$ is a positively oriented, simple, closed, smooth, $G$-contractible curve such that $w$ is inside $\gamma$. Then

$$f'(w) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{(z-w)^2} \, dz$$

and

$$f''(w) = \frac{1}{\pi i} \int_{\gamma} \frac{f(z)}{(z-w)^3} \, dz.$$  

This innocent-looking theorem has a very powerful consequence: just from knowing that $f$ is holomorphic we know of the existence of $f''$, that is, $f'$ is also holomorphic in $G$. Repeating this argument for $f'$, then for $f''$, $f'''$, etc., gives the following statement, which has no analog whatsoever in the reals.

**Corollary 5.2.** If $f$ is differentiable in the region $G$ then $f$ is infinitely differentiable in $G$.

**Proof of Theorem 1.1.** The idea of the proof is very similar to the proof of Cauchy’s integral formula (Theorem 4.10). We will study the following difference quotient, which we can rewrite as follows by Theorem 4.10.

$$\frac{f(w + \Delta w) - f(w)}{\Delta w} = \frac{1}{\Delta w} \left( \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - (w + \Delta w)} \, dz - \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - w} \, dz \right)$$

$$= \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{(z - w - \Delta w)(z - w)} \, dz.$$
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Hence we will have to show that the following expression gets arbitrarily small as \( \Delta w \to 0 \):

\[
\frac{f(w + \Delta w) - f(w)}{\Delta w} - \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{(z - w)^2} \, dz = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{(z - w - \Delta w)(z - w)} - \frac{f(z)}{(z - w)^2} \, dz
\]

\[
= \Delta w \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{(z - w - \Delta w)(z - w)^2} \, dz.
\]

This can be made arbitrarily small if we can show that the integral stays bounded as \( \Delta w \to 0 \). In fact, by Proposition 4.4(d), it suffices to show that the integrand stays bounded as \( \Delta w \to 0 \) (because \( \gamma \) and hence \( \text{length}(\gamma) \) are fixed). Let \( M = \max_{z \in \gamma} |f(z)| \) and \( N = \max_{z \in \gamma} |z - w| \). Since \( \gamma \) is a closed set, there is some positive \( \delta \) so that the open disk of radius \( \delta \) around \( w \) does not intersect \( \gamma \); that is, \( |z - w| \geq \delta \) for all \( z \) on \( \gamma \). By the reverse triangle inequality we have for all \( z \in \gamma \)

\[
\left| \frac{f(z)}{(z - w - \Delta w)(z - w)^2} \right| \leq \frac{|f(z)|}{(|z - w| - |\Delta w|)|z - w|^2} \leq \frac{M}{(\delta - |\Delta w|)N^2},
\]

which certainly stays bounded as \( \Delta w \to 0 \). The proof of the formula for \( f'' \) is very similar and will be left for the exercises (see Exercise 2).

Remarks. 1. Theorem 1.1 suggests that there are similar formulas for the higher derivatives of \( f \). This is in fact true, and theoretically one could obtain them one by one with the methods of the proof of Theorem 1.1. However, once we start studying power series for holomorphic functions, we will obtain such a result much more easily; so we save the derivation of formulas for higher derivatives of \( f \) for later (see Corollary 8.6).

2. Theorem 1.1 can also be used to compute certain integrals. We give some examples of this application next.

Example 5.3. \[
\int_{|z|=1} \frac{\sin(z)}{z^2} \, dz = 2\pi i \left. \frac{d}{dz} \sin(z) \right|_{z=0} = 2\pi i \cos(0) = 2\pi i.
\]

Example 5.4. To compute the integral

\[
\int_{|z|=2} \frac{dz}{z^2(z - 1)},
\]

we first split up the integration path as illustrated in Figure 1.1: Introduce an additional path which separates 0 and 1. If we integrate on these two new closed paths (\( \gamma_1 \) and \( \gamma_2 \)) counterclockwise, the two contributions along the new path will cancel each other. The effect is that we transformed an integral, for which two singularities where inside the integration path, into a sum of two integrals, each of which has only one singularity inside the integration path; these new integrals we know
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Figure 5.1: Example 1.4

how to deal with.

\[ \int_{|z|=2} \frac{dz}{z^2(z-1)} = \int_{\gamma_1} \frac{dz}{z^2(z-1)} + \int_{\gamma_2} \frac{dz}{z^2(z-1)} \]

\[ = \int_{\gamma_1} \frac{1}{z^2} \, dz + \int_{\gamma_2} \frac{1}{z-1} \, dz \]

\[ = 2\pi i \left( \frac{d}{dz} \frac{1}{z-1} \right) \bigg|_{z=0} + 2\pi i \frac{1}{1^2} \]

\[ = 2\pi i \left( -\frac{1}{(-1)^2} \right) + 2\pi i \]

\[ = 0. \]

Example 5.5.

\[ \int_{|z|=1} \frac{\cos(z)}{z^3} \, dz = \pi i \frac{d^2}{dz^2} \cos(z) \bigg|_{z=0} = \pi i (-\cos(0)) = -\pi i. \]

5.2 Taking Cauchy’s Formula to the Limit

Many beautiful applications of Cauchy’s formula arise from considerations of the limiting behavior of the formula as the curve gets arbitrarily large. We shall look at a few applications along these lines in this section, but this will be a recurring theme throughout the rest of the book.

The first application is understanding the roots of polynomials. As a preparation we prove the following inequality, which is generally quite useful. It simply says that for large enough \( z \), a polynomial of degree \( d \) looks almost like a constant times \( z^d \).

Lemma 5.6. Suppose \( p(z) \) is a polynomial of degree \( d \) with leading coefficient \( a_d \). Then there is real number \( R_0 \) so that

\[ \frac{1}{2} |a_d| |z|^d \leq |p(z)| \leq 2 |a_d| |z|^d \]

for all \( z \) satisfying \( |z| \geq R_0 \).
Proof. Since \( p(z) \) has degree \( d \) its leading coefficient \( a_d \) is not zero, and we can factor out \( a_d z^d \):

\[
|p(z)| = \left| a_d z^d + a_{d-1} z^{d-1} + a_{d-2} z^{d-2} + \cdots + a_1 z + a_0 \right|
\]

\[
= |a_d| |z|^d \left| 1 + \frac{a_{d-1}}{a_d z} + \frac{a_{d-2}}{a_d z^2} + \cdots + \frac{a_1}{a_d z^{d-1}} + \frac{a_0}{a_d z^d} \right|.
\]

Then the sum inside the last factor has limit 1 as \( z \to \infty \) so its modulus is between \( \frac{1}{2} \) and 2 for all large enough \( z \).

Theorem 5.7 (Fundamental Theorem of Algebra\(^1\)). Every non-constant polynomial has a root in \( \mathbb{C} \).

Proof.\(^2\) Suppose (by way of contradiction) that \( p \) does not have any roots, that is, \( p(z) \neq 0 \) for all \( z \in \mathbb{C} \). Then Cauchy’s formula gives us

\[
\frac{1}{p(0)} = \frac{1}{2\pi i} \int_{C_R} \frac{1/p(z)}{z} \, dz
\]

where \( C_R \) is the circle of radius \( R \) around the origin. Notice that the value of the integral does not depend on \( R \), so we have

\[
\frac{1}{p(0)} = \lim_{R \to \infty} \frac{1}{2\pi i} \int_{C_R} \frac{dz}{zp(z)}.
\]

But now we can see that the limit of the integral is 0: By Lemma 1.6 we have \( |z p(z)| \geq \frac{1}{2} |a_d| |z|^{d+1} \) for all large \( z \), where \( d \) is the degree of \( p(z) \) and \( a_d \) is the leading coefficient of \( p(z) \). Hence, using Proposition 4.4(d) and the formula for the circumference of a circle we see that the integral can be bounded as

\[
\left| \frac{1}{2\pi i} \int_{C_R} \frac{dz}{zp(z)} \right| \leq \frac{1}{2\pi} \cdot \frac{2}{|a_d| R^{d+1}} \cdot (2\pi R) = \frac{2}{|a_d|} R^d
\]

and this has limit 0 as \( R \to \infty \). But, plugging into (*), we have shown that \( \frac{1}{p(0)} = 0 \), which is impossible. \( \square \)

Remarks. 1. This statement implies that any polynomial \( p \) can be factored into linear terms of the form \( z - a \) where \( a \) is a root of \( p \), as we can apply the corollary, after getting a root \( a \), to \( \frac{p(z)}{z-a} \) (which is again a polynomial by the division algorithm), etc. (see also Exercise 11).

2. A compact reformulation of the Fundamental Theorem of Algebra is to say that \( \mathbb{C} \) is algebraically closed. Thus, \( \mathbb{R} \) is not algebraically closed.

Example 5.8. The polynomial \( p(x) = 2x^4 + 5x^2 + 3 \) is such that all of its coefficients are real. However, \( p \) has no roots in \( \mathbb{R} \). The Fundamental Theorem of Algebra states that \( p \) must have one (in fact, 4) roots in \( \mathbb{C} \):

\[
p(x) = (x^2 + 1)(2x^2 + 3) = (x + i)(x - i)(\sqrt{2}x + \sqrt{3}i)(\sqrt{2}x - \sqrt{3}i).
\]

\(^1\)The Fundamental Theorem of Algebra was first proved by Gauß (in his doctoral dissertation), although its statement had been assumed to be correct long before Gauß’s time.

\(^2\)It is amusing that such an important algebraic result can be proved ‘purely analytically.’ There are proofs of the Fundamental Theorem of Algebra which do not use complex analysis. On the other hand, as far as we are aware, all proofs use some analysis (such as the intermediate-value theorem).
A powerful consequence of (the first half of) Theorem 1.1 is the following.

**Corollary 5.9** (Liouville’s Theorem). Every bounded entire function is constant.

*Proof.* Suppose \(|f(z)| \leq M\) for all \(z \in \mathbb{C}\). Given any \(w \in \mathbb{C}\), we apply Theorem 1.1 with the circle \(C_R\) of radius \(R\) centered at \(w\). Note that we can choose any \(R\) because \(f\) is entire. Now we apply Proposition 4.4 (d), remembering that \(C_R\) has circumference \(2\pi R\) and \(|z-w| = R\) for all \(z\) on \(C_R\):

\[
|f'(w)| = \left| \frac{1}{2\pi i} \int_{C_R} \frac{f(z)}{(z-w)^2} \, dz \right| \leq \frac{1}{2\pi} \max_{z \in \gamma_R} \frac{|f(z)|}{|z-w|^2} \cdot 2\pi R = \frac{1}{2\pi} \max_{z \in \gamma_R} \frac{|f(z)|}{R^2} 2\pi R = \frac{\max |f(z)|}{R},
\]

where \(\gamma_R\) is the unit circle centered at \(z\). The right-hand side can be made arbitrary small, as we are allowed to make \(R\) as large as we want. This implies that \(f' = 0\), and hence, by Theorem 2.15, \(f\) is constant.

As an example of the usefulness of Liouville’s theorem we give another proof of the fundamental theorem of algebra, which is close to Gauß’s original proof:

*Another proof of the fundamental theorem of algebra.* Suppose (by way of contradiction) that \(p\) does not have any roots, that is, \(p(z) \neq 0\) for all \(z \in \mathbb{C}\). Then, because \(p\) is entire, the function \(f(z) = \frac{1}{p(z)}\) is entire. But \(f \to 0\) as \(|z|\) becomes large as a consequence of Lemma 1.6; that is, \(f\) is also bounded (Exercise 10). Now apply Corollary 1.9 to deduce that \(f\) is constant. Hence \(p\) is constant, which contradicts our assumptions.

As one more example of this theme of getting results from Cauchy’s formula by taking the limit as a path goes to infinity, we compute an improper integral.

**Example 5.10.** Let \(\sigma\) be the counterclockwise semicircle formed by the segment \(S\) of the real axis from \(-R\) to \(R\), followed by the circular arc \(T\) of radius \(R\) in the upper half plane from \(R\) to \(-R\), where \(R > 1\). We shall integrate the function

\[
f(z) = \frac{1}{z^2 + 1} = \frac{1/(z+i)}{z-i} = \frac{g(z)}{z-i},\]

where \(g(z) = \frac{1}{z+i}\)

Since \(g(z)\) is holomorphic inside and on \(\sigma\) and \(i\) is inside \(\sigma\), we can apply Cauchy’s formula:

\[
\frac{1}{2\pi i} \int_{\sigma} \frac{dz}{z^2 + 1} = \frac{1}{2\pi i} \int_{\sigma} \frac{g(z)}{z-i} \, dz = g(i) = \frac{1}{i + i} = \frac{1}{2i},
\]

and so

\[
\int_{S} \frac{dz}{z^2 + 1} + \int_{T} \frac{dz}{z^2 + 1} = \int_{\sigma} \frac{dz}{z^2 + 1} = 2\pi i \cdot \frac{1}{2i} = \pi. \tag{**}
\]

Now this formula holds for all \(R > 1\), so we can take the limit as \(R \to \infty\). First, \(|z^2 + 1| \geq \frac{1}{2} |z|^2\) for large enough \(z\) by Lemma 1.6, so we can bound the integral over \(T\) using Proposition 4.4(d):

\[
\left| \int_{T} \frac{dz}{z^2 + 1} \right| \leq \frac{2}{R^2} \cdot \pi R = \frac{2}{R}
\]

\(^3\)For more information about Joseph Liouville (1809–1882), see [http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Liouville.html](http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Liouville.html).

\(^4\)This theorem is for historical reasons erroneously attributed to Liouville. It was published earlier by Cauchy; in fact, Gauß may well have known about it before Cauchy.
and this has limit 0 as $R \to \infty$. On the other hand, we can parameterize the integral over $S$ using $z = t$, $-R \leq t \leq R$, obtaining
\[
\int_S \frac{dz}{z^2 + 1} = \int_{-R}^{R} \frac{dt}{1 + t^2}.
\]
As $R \to \infty$ this approaches an improper integral. Making these observations in the limit of the formula (***) as $R \to \infty$ now produces
\[
\int_{-\infty}^{\infty} \frac{dt}{t^2 + 1} = \pi.
\]
Of course this integral can be evaluated almost as easily using standard formulas from calculus. However, just a slight modification of this example leads to an improper integral which is far beyond the scope of basic calculus; see Exercise 14.

5.3 Antiderivatives

We begin this section with a familiar definition from real calculus:

**Definition 5.11.** Let $G$ be a region of $\mathbb{C}$. For any functions $f, F : G \to \mathbb{C}$, if $F$ is holomorphic on $G$ and $F'(z) = f(z)$ for all $z \in G$, then $F$ is an antiderivative of $f$ on $G$, also known as a primitive of $f$ on $G$.

In short, an antiderivative of $f$ is a function with $F' = f$.

**Example 5.12.** We have already seen that $F(z) = z^2$ is entire, and has derivative $f(z) = 2z$. Thus, $F$ is an antiderivative of $f$ on any region $G$.

Just like in the real case, there are complex versions of the Fundamental Theorems of Calculus. The Fundamental Theorems of Calculus makes a number of important claims: that continuous functions are integrable, their antiderivatives are continuous and differentiable, and that antiderivatives provide easy ways to compute values of definite integrals. The difference between the real case and the complex case is that for the complex case, we need to think about integrals over arbitrary curves and 2-dimensional regions.

To state the first Fundamental Theorem, we need some topological definitions:

**Definition 5.13.** A region $G \subset \mathbb{C}$ is simply connected if every simply closed curve in $G$ is $G$-contractible. That is, for any simple closed curve $\gamma \subset G$, the interior of $\gamma$ in $\mathbb{C}$ is also completely contained in $G$.

Loosely, simply connected means $G$ has no ‘holes’.

**Theorem 5.14.** (The First Fundamental Theorem of Calculus) Suppose $G \subset \mathbb{C}$ is a simply-connected region, and fix some basepoint $z_0 \in G$. For each point $z \in G$, let $\gamma_z$ denote a smooth curve in $G$ from $z_0$ to $z$. Let $f : G \to \mathbb{C}$ be a holomorphic function. Then the function $F(z) : G \to \mathbb{C}$ defined by
\[
F(z) := \int_{\gamma_z} f
\]
is holomorphic on $G$ with $F'(z) = f(z)$.
In short, every holomorphic function on a simply-connected region has a primitive.

Proof. We leave this to the exercises, Exercise 15. □

Theorem 5.15. [The Second Fundamental Theorem of Calculus] Suppose \( G \subseteq \mathbb{C} \) is a simply connected region. Let \( \gamma \subset G \) be a smooth curve with parametrization \( \gamma(t), a \leq t \leq b \). If \( f : G \to \mathbb{C} \) is holomorphic on \( G \) and \( F \) is any primitive of \( f \) on \( G \), then

\[
\int_{\gamma} f = F(\gamma(b)) - F(\gamma(a)).
\]

Remarks. 1. Actually, more is true. The assumptions that \( G \) is simply connected and \( f \) is holomorphic are both unnecessary.

Proof. The antiderivative \( F \) prescribed by the First Fundamental Theorem of Calculus satisfies the desired equation by definition. For any other antiderivative \( G \) of \( f \), we have that \( F'(z) = G'(z) \) for \( z \in G \), so the function \( H(z) := F(z) - G(z) \) is holomorphic with derivative 0, so is constant. Thus, \( G(z) = F(z) + c \) for some constant \( c \in \mathbb{C} \). Then

\[
G(\gamma(b)) - G(\gamma(a)) = F(\gamma(b)) - F(\gamma(a)) = \int_{\gamma} f,
\]

as desired. □

There are many interesting consequences of the Fundamental Theorems. We begin with two consequences of the First Fundamental Theorem. Because the primitive \( F \) of a function \( f \) on a region \( G \) is by definition differentiable on \( G \), the primitive \( F \) itself has a primitive on \( G \), which also has a primitive, which also has a primitive, etc. Thus, we may go ‘in the other direction’ from Corollary 1.2:

Corollary 5.16. If \( f \) is differentiable in the region \( G \) then \( f \) is infinitely integrable in \( G \).

Another consequence comes from the proof of Theorem 1.14: we did not really need the fact that every closed curve in \( G \) is contractible, just that every closed curve gives a zero integral for \( f \). This fact can be exploited to give a sort of converse statement to Corollary 4.7.

Corollary 5.17 (Morera’s\(^5\) Theorem). Suppose \( f \) is continuous in the region \( G \) and

\[
\int_{\gamma} f = 0
\]

for all smooth closed paths \( \gamma \subset G \). Then \( f \) is holomorphic in \( G \).

Proof. As in the proof of Theorem 1.14, we fix an \( a \in G \) and define

\[
F(z) = \int_{\gamma_a} f,
\]

where \( \gamma_a \) is any smooth curve in \( G \) from \( a \) to \( z \). As above, this is a well-defined function because all closed paths give a zero integral for \( f \); and exactly as above we can show that \( F \) is a primitive for \( f \) in \( G \). Because \( F \) is holomorphic on \( G \), Corollary 1.2 gives that \( F \) is also holomorphic on \( G \). □

\(^5\)For more information about Giancinto Morera (1856–1907), see http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Morera.html.
We now mention two interesting corollaries of the Second Fundamental Theorem.

**Corollary 5.18.** If $f$ is holomorphic on $G$, then an antiderivative of $f$ exists on $G$, and $\int_\gamma f$ is independent of the path $\gamma \subset G$ between $\gamma(a)$ and $\gamma(b)$.

If $f$ is holomorphic on $G$, we say $\int_\gamma f$ is *path-independent*.

Example 4.2 shows that a path-independent integral is quite special; it also says that the function $z^2$ does not have an antiderivative in, for example, the region $\{z \in \mathbb{C} : |z| < 2\}$. (Actually, the function $z^2$ does not have an antiderivative in any nonempty region—prove it!)

In the special case that $\gamma$ is closed (that is, $\gamma(a) = \gamma(b)$), we immediately get the following nice consequence (which also follows from Cauchy’s Integral Formula).

**Corollary 5.19.** Suppose $G \subseteq \mathbb{C}$ is open, $\gamma$ is a smooth closed curve in $G$, and $f$ is holomorphic on $G$ and has an antiderivative on $G$. Then

$$\int_\gamma f = 0.$$ 

**Exercises**

1. Compute the following integrals, where $C$ is the boundary of the square with corners at $\pm 4 \pm 4i$:
   
   (a) $\int_C \frac{e^z}{z^3} \, dz$.
   
   (b) $\int_C \frac{e^z}{(z - \pi i)^2} \, dz$.
   
   (c) $\int_C \frac{\sin(2z)}{(z - \pi)^2} \, dz$.
   
   (d) $\int_C \frac{e^z \cos(z)}{(z - \pi)^3} \, dz$.

2. Prove the formula for $f''$ in Theorem 1.1.

3. Integrate the following functions over the circle $|z| = 3$, oriented counterclockwise:

   (a) $\log(z - 4i)$.
   
   (b) $\frac{1}{z - \frac{1}{2}}$.
   
   (c) $\frac{1}{z^2 - 4}$.
   
   (d) $\exp(z)$.
   
   (e) $\left(\frac{\cos z}{z}\right)^2$.
   
   (f) $i^z - 3$.
   
   (g) $\frac{\sin z}{(z^2 + \frac{1}{2})^2}$.
   
   (h) $\frac{\exp z}{(z - w)^2}$, where $w$ is any fixed complex number with $|w| \neq 3$. 

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(i) \( \frac{1}{(z+4)(z^2+1)}. \)

4. Compute \( \int_{|z|=1} \frac{dz}{z^p} \) for an arbitrary integer \( p \), by using the change of variables \( w = \frac{1}{z} \) when appropriate.

5. Evaluate \( \int_{|z|=3} \frac{e^{2z}dz}{(z-1)^2(z-2)}. \)

6. Prove that \( \int_{\gamma} z \exp(z^2) \, dz = 0 \) for any closed curve \( \gamma \).

7. Show that \( \exp(\sin z) \) has an antiderivative on \( \mathbb{C} \).

8. Find a (maximal size) set on which \( f(z) = \exp\left(\frac{1}{z}\right) \) has an antiderivative. (How does this compare with the real function \( f(x) = e^{1/x} \)?)

9. Compute the following integrals; use the principal value of \( z^i \). (Hint: one of these integrals is considerably easier than the other.)
   
   (a) \( \int_{\gamma_1} z^i \, dz \) where \( \gamma_1(t) = e^{it}, \ -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}. \)

   (b) \( \int_{\gamma_2} z^i \, dz \) where \( \gamma_2(t) = e^{it}, \ \frac{\pi}{2} \leq t \leq \frac{3\pi}{2}. \)

10. Suppose \( f \) is continuous on \( \mathbb{C} \) and \( \lim_{z \to \infty} f(z) = 0 \). Show that \( f \) is bounded. (Hint: From the definition of limit at infinity (with \( \epsilon = 1 \)) there is \( R > 0 \) so that \( |f(z) - 0| = |f(z)| < 1 \) if \( |z| > R \). Is \( f \) bounded for \( |z| \leq R? \)

11. Let \( p \) be a polynomial of degree \( n > 0 \). Prove that there exist complex numbers \( c, z_1, z_2, \ldots, z_k \) and positive integers \( j_1, \ldots, j_k \) such that

   \[ p(z) = c(z - z_1)^{j_1}(z - z_2)^{j_2} \cdots (z - z_k)^{j_k}, \]

   where \( j_1 + \cdots + j_k = n \).

12. Show that a polynomial of odd degree with real coefficients must have a real zero. (Hint: Exercise 20b in Chapter 1.)

13. Suppose \( f \) is entire and there exist constants \( a, b \) such that \( |f(z)| \leq a|z| + b \) for all \( z \in \mathbb{C} \). Prove that \( f \) is a linear polynomial (that is, of degree \( \leq 1 \)).

14. In this problem \( F(z) = \frac{e^{iz}}{x^2+1} \) and \( R > 1 \). Modify the example at the end of Section 1.2:

   (a) Show that \( \int_{\sigma} F(z) \, dz = \frac{2\pi}{2} \) if \( \sigma \) is the counterclockwise semicircle formed by the segment \( S \) of the real axis from \(-R\) to \( R \), followed by the circular arc \( T \) of radius \( R \) in the upper half plane from \( R \) to \(-R\).

   (b) Show that \( |e^{iz}| \leq 1 \) for \( z \) in the upper half plane, and conclude that \( |F(z)| \leq \frac{2\pi}{|z|^2} \) for \( z \) large enough.

   (c) Show that \( \lim_{R \to \infty} \int_{T} F(z) \, dz = 0 \), and hence \( \lim_{R \to \infty} \int_{S} F(z) \, dz = \frac{\pi}{\epsilon}. \)

   (d) Conclude, by parameterizing the integral over \( S \) in terms of \( t \) and just considering the real part, that \( \int_{-\infty}^{\infty} \frac{\cos(t)}{x^2+1} \, dx = \frac{\pi}{\epsilon}. \)
15. Prove Theorem 1.14, as follows.

(a) Use Cauchy’s Theorem to show that, for a given \( z \in G \), the value of \( F(z) \) is independent of the choice of \( \gamma_z \).

(b) Fix \( z, z' \in G \) such that the straight line \( \gamma \) connecting \( z \) to \( z' \) is contained in \( G \). Again using Cauchy’s Theorem, show that

\[
F(z') - F(z) = \int_{\gamma} f.
\]

(c) Use the fact that \( f \) is continuous to show that for any fixed \( z \in \mathbb{C} \) and any \( \epsilon > 0 \), there is a \( \Delta z \in \mathbb{C} \) such that

\[
\left| \frac{F(z) - F(z + \Delta z)}{\Delta z} - f(z) \right| < \epsilon.
\]

(d) Conclude that \( F'(z) = f(z) \).