WAVELETS, WAVELET SETS, AND LINEAR ACTIONS ON $\mathbb{R}^n$

GESTUR ÓLAFSSON AND DARRIN SPEEGLE

Abstract. Wavelet and frames have become a widely used tool in mathematics, physics, and applied science during the last decade. This article gives an overview over some well known results about the continuous and discrete wavelet transforms and groups acting on $\mathbb{R}^n$. We also show how this action can give rise to wavelets, and in particular, MSF wavelets in $L^2(\mathbb{R}^n)$.

Introduction

The classical wavelet system consists of a single function $\psi \in L^2(\mathbb{R})$ such that $\{2^{j/2}\psi(2^jx+k) \mid j,k \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{R})$. There has been quite a bit of recent interest in relaxing various aspects of the definition of wavelets, in particular in higher dimensions. For example, one can allow multiple functions $\psi^i, \ldots, \psi^L$, an arbitrary matrix of dilations, and an arbitrary lattice of translations. One could relax even further to allow a group of dilations, or perhaps even just a set of dilations and translations. A first question one would ask, then, is: for which collections of dilations and translations do there exist wavelets? We will begin by reviewing some well-known results concerning this central question. Then, we will show that there is a fundamental connection between the papers of Dai, Diao, Gu and Han [15], Fabec and Ólafsson [21], Laugesen, Weaver, Weiss and Wilson [46], and Wang [57]. One argument that a survey paper such as this one is useful is that, even though these eleven authors are active in the field, there is only one cross-reference of the above papers in the references of the other papers.

We now describe briefly the connection between the papers listed above. All four papers are concerned with constructing reproducing systems consisting of dilations and translations of a function. That is, they consider triples $(\mathcal{D}, \mathcal{T}, M)$, where $\mathcal{D}$ is some collection of invertible matrices, $\mathcal{T}$ is some collection of points in $\mathbb{R}^n$, and M a non-trivial closed subspace of $L^2(\mathbb{R}^n)$. Then, they ask whether there is a function ψ such that $\{\psi_{a,k} = |\det a|^{1/2}\,\psi(a(x)+k) \mid a \in \mathcal{D}, k \in \mathcal{T}\}$ is a frame, normalized tight frame, or even an orthonormal basis for M

In [15], it is assumed that $\mathcal{D} = \{a^j \mid j \in \mathbb{Z}\}$ for some expansive matrix a, that $\mathcal{T} = \mathbb{Z}^n$, and that M is an a-invariant subspace of $L^2(\mathbb{R}^n)$. In [21], the assumptions are that, $\mathcal{D}$ is constructed as a subset of a particular type of group H, that $\mathcal{T}$ is a full rank lattice depending on H, and finally that M is of the form $M = \{f \in L^2(\mathbb{R}^n) \mid \mathrm{Supp}(\hat{f}) \subseteq \overline{\mathcal{O}}\}$, where $\mathcal{O} \subset \mathbb{R}^n$ is an open H-orbit. In [46], it is assumed that $\mathcal{D}$ is a group, $\mathcal{T} = \mathbb{Z}^n$, and $M = L^2(\mathbb{R}^n)$. In [57], it is assumed that $\mathcal{D}$ and $\mathcal{T}$ satisfy non-algebraic conditions relating to the existence of fundamental regions (see Section 1 for details) and $M = L^2(\mathbb{R}^n)$. Moreover, all four papers - either explicitly or implicitly - are concerned primarily with the existence of functions of the form $\hat{\psi} = \chi_\Omega$.

When put in this general framework, it becomes clear that the four papers are related in spirit and scope. What we will show below is that they are also related in that results in [15] can be used to remove technical assumptions from results in [57]. The improved results in [57] can then be used to improve the results in [46] and [21]. We will improve the results in [21] by removing the dependence of the lattice on the group, and by constructing an orthonormal basis where a normalized tight frame was constructed

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before. The proof of the main Theorem in [21] will also be simplified. Finally, we improve the results in [46] by replacing normalized tight frame system with a wavelet system.

We will attempt to make these technical improvements to the theorems in these papers with a minimal amount of technical work. In particular, where possible, we will apply theorem quoting proofs. The primary exception to this is Theorem 1.18, where we essentially need to check that the details of an argument in [18] go through in a slightly more general setting.

1. Wavelet sets

We start this section by recalling some simple definitions and facts about wavelets, wavelet sets, and tilings. For a measurable set $\Omega \subseteq \mathbb{R}^n$ we denote by χ_Ω the indicator function of the set Ω and by $|\Omega| = \int \chi_\Omega(x)\,dx$ the measure of Ω.

Definition 1.1. Let (M,μ) be a measure space. A countable collection $\{\Omega_j\}$ of subsets of M is a (measurable) tiling of M if $\mu(M \setminus \bigcup_j \Omega_j) = 0$, and $\mu(\Omega_i \cap \Omega_j) = 0$ for $i \neq j$.

Definition 1.2. Let $\mathcal{T} \subset \mathbb{R}^n$ and $\mathcal{D} \subset \mathrm{GL}(n,\mathbb{R})$. We say that $\mathcal{D}$ is a *multiplicative tiling set* of $\mathbb{R}^n$ if there exists a set $\Omega \subset \mathbb{R}^n$ of positive measure such that $\{d\Omega \mid d \in \mathcal{D}\}$ is a tiling of $\mathbb{R}^n$. The set Ω is said to be a *multiplicative $\mathcal{D}$-tile*. We say $\mathcal{D}$ is a *bounded multiplicative tiling set* of $\mathbb{R}^n$ if there is a multiplicative $\mathcal{D}$-tile Ω which is bounded and such that $0 \notin \overline{\Omega}$.

Similarly, we say that $\mathcal{T}$ is an *additive tiling set* of $\mathbb{R}^n$ if there exists a set $\Omega \subset \mathbb{R}^n$ such that $\{\Omega + x \mid x \in \mathcal{T}\}$ is a tiling of $\mathbb{R}^n$. The set Ω is said to be an *additive $\mathcal{T}$-tile*. Again, we add the word bounded if Ω can be chosen to be a bounded set (with no restriction on being bounded away from 0).

A set Ω is a *$(\mathcal{D},\mathcal{T})$-tiling set* if it is a $\mathcal{D}$ multiplicative tiling set and a $\mathcal{T}$ additive tiling set.

Note that this definition does not coincide with the definition of Wang [57]. Wang defines a multiplicative tiling set to be what we have defined to be a bounded multiplicative tiling set. We feel that boundedness properties of $\mathcal{D}$-tiles are interesting properties, but they should not be part of a definition of tiling.

Multiplicative and additive tilings of $\mathbb{R}^n$ show up in wavelet theory and other branches of analysis in a natural way.

Definition 1.3. A function $\varphi \in L^2(\mathbb{R}^n)$ is called a *wavelet* if there exists a subset $\mathcal{D} \subset \mathrm{GL}(n,\mathbb{R})$ and a subset $\mathcal{T} \subset \mathbb{R}^n$ such that

$$\mathcal{W}(\varphi;\mathcal{D},\mathcal{T}) := \{|\det d|^{1/2}\varphi(dx+k) \mid d \in \mathcal{D},\, k \in \mathcal{T}\}$$

forms an orthonormal basis for $L^2(\mathbb{R}^n)$. The set $\mathcal{D}$ is called the *dilation set* for φ, the set $\mathcal{T}$ is called the *translation set* for φ, and we say that φ is a *$(\mathcal{D},\mathcal{T})$-wavelet.*

Normalize the Fourier transform by

$$\mathcal{F}(f)(\lambda) = \hat{f}(\lambda) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i(\lambda,x)}\,dx\,.$$

We set $f^\vee(x) = \hat{f}(-x)$. Then $f = (\hat{f})^\vee$. For simplicity we set $e_\lambda(x) = e^{2\pi i(\lambda,x)}$.

Definition 1.4. Let $\Omega \subset \mathbb{R}^n$ be measurable with positive, but finite measure. We say that Ω is *a wavelet set* if there exists a pair $(\mathcal{D},\mathcal{T})$, with $\mathcal{D} \subset \mathrm{GL}(n,\mathbb{R})$ and $\mathcal{T} \subset \mathbb{R}^n$ such that $\chi_\Omega^\vee$ is a $(\mathcal{D},\mathcal{T})$-wavelet. If $\chi_\Omega^\vee$ is a $(\mathcal{D},\mathcal{T})$-wavelet, then we say that Ω is a *$(\mathcal{D},\mathcal{T})$-wavelet set.*

Definition 1.5. A measurable set $\Omega \subset \mathbb{R}^n$ with finite positive measure is called *a spectral set* if there exists a set $\mathcal{T} \subset \mathbb{R}^n$ such that the sequence of functions $\{e_\lambda\}_{\lambda\in\mathcal{T}}$ forms an orthogonal basis for $L^2(\Omega)$. If this is the case we say that $\mathcal{T}$ is the *spectrum of* Ω, and say that $(\Omega,\mathcal{T})$ is a spectral pair

Now, after given this list of definitions, let us recall some results, questions, and conjectures on how these concepts are tied together. A first result, which has appeared in several places [16, 36, 40] is

Theorem 1.6. *A measurable set $\Omega \subset \mathbb{R}$ is a wavelet set for the pair $\mathcal{D} = \{2^n \mid n \in \mathbb{Z}\}$ and $\mathcal{T} = \mathbb{Z}$ if and only if Ω is a $(\mathcal{D}, \mathcal{T})$-tiling set.*

The proof of Theorem 1.6 given in [16], for example, works to prove the following

Theorem 1.7. *Let a be an invertible matrix. A measurable set $\Omega \subset \mathbb{R}^n$ is a wavelet set for the pair $\mathcal{D} = \{a^n \mid n \in \mathbb{Z}\}$ and the full rank lattice $\mathcal{T}$ if and only if Ω is a $(\mathcal{D}, \mathcal{T})$-tiling set.*

For the general case we have now the following two related questions:

Question 1 (Wang, [57])**.** For which sets $\mathcal{D} \subset \mathrm{GL}(n, \mathbb{R})$, $\mathcal{T} \subset \mathbb{R}^n$ do there exist $(\mathcal{D}, \mathcal{T})$-wavelets?

Question 2. For which sets $\mathcal{D} \subset \mathrm{GL}(n, \mathbb{R})$, $\mathcal{T} \subset \mathbb{R}^n$ do there exist $(\mathcal{D}, \mathcal{T})$-wavelet sets?

Clearly, if there exists a $(\mathcal{D}, \mathcal{T})$-wavelet set, then there exists a $(\mathcal{D}, \mathcal{T})$-wavelet, but, what is interesting, is that the converse may also be true. In particular, there are currently no examples known of sets $(\mathcal{D}, \mathcal{T})$ for which there exist wavelets, but for which there do not exist $(\mathcal{D}, \mathcal{T})$-wavelet sets. Therefore we can state the third natural question:

Question 3. Is it true that if there exists a $(\mathcal{D}, \mathcal{T})$-wavelet, then there exists a $(\mathcal{D}, \mathcal{T})$-wavelet set?

So far, all evidence points to a positive answer for question 3. (Though, we should point out that question 3 has mostly been thought about in the case that $\mathcal{D}$ is a singly generated group and $\mathcal{T}$ is a full rank lattice, so it is possible that there is a relatively easy counterexample to the question posed in this generality.) When $\mathcal{D}$ is generated by a single matrix a and $\mathcal{T}$ is a full rank lattice, it is known [17] that if a is expansive, then there exist $(\mathcal{D}, \mathcal{T})$-wavelet sets. Moreover, it is also known [11, 13, 14] in the expansive case that there exist $(\mathcal{D}, \mathcal{T})$-wavelets that do not come from a wavelet set if and only if there is a $j \neq 0$ such that $(a^T)^j(\mathcal{T}^*) \cap \mathcal{T}^* \neq \{0\}$, where $\mathcal{T}^*$ is the dual lattice defined by $\mathcal{T}^* = \{\alpha \in \mathbb{R}^n :< \alpha, \beta >\in \mathbb{Z}$ for all $\beta \in \mathcal{T}\}$. In particular, for *most* pairs of this type, the *only* wavelets that exist come from wavelet sets. When $\mathcal{D}$ is generated by a not necessarily expansive matrix a and $\mathcal{T}$ is a lattice, then the handful of $(\mathcal{D}, \mathcal{T})$-wavelets known all come from $(\mathcal{D}, \mathcal{T})$-wavelet sets.

There is also a stronger version of question 3 due to Larson [45] in the one dimensional case.

Question 4 (Larson, [45])**.** Is it true that if ψ is a $(\mathcal{D}, \mathcal{T})$-wavelet, then there is a $(\mathcal{D}, \mathcal{T})$-wavelet set $E \subseteq \mathrm{supp}(\hat{\psi})$?

This problem is open even for the "classical" case of dimension 1 with dilations by powers of 2 and translations by integers. We name two partial answers. The first is given by Rzeszotnik in his PhD Thesis, and the second is due to Rzeszotnik and the second author of this paper.

Theorem 1.8 (Rzeszotnik, [52] Corollary 3.10)**.** *Every multiresolution analysis (MRA) $(2^j, \mathbb{Z})$-wavelet contains in the support of its Fourier transform an MRA $(\mathcal{D}, \mathcal{T})$-wavelet set.*

Theorem 1.9. [53] *If ψ is a classical wavelet and the set $K = \mathrm{supp}(\hat{\psi})$ satisfies*

1. $\sum_{k\in\mathbb{Z}} \chi_K(\xi + k) \leq 2$ *a.e.;*
2. $\sum_{k\in\mathbb{Z}} \chi_K(2^j\xi) \leq 2$ *a.e.*

Then K contains a wavelet set.

Qing Gu has an unpublished example which shows that the techniques in [53] do not extend to the case that $\sum_{k\in\mathbb{Z}} \chi_K(\xi + k) \leq 3$ a.e. and $\sum_{j\in\mathbb{Z}} \chi_K(2^j\xi) \leq 3$ a.e.

Tilings and spectral sets are related by the *Fuglede conjecture* [27]

Conjecture 1 (Fuglede)**.** *A measurable set Ω, with positive and finite measure is a spectral set if and only if Ω is an additive $\mathcal{T}$-tile for some set $\mathcal{T}$.*

The conjecture, in general, still remains unsolved, even if several partial results have been obtained [41, 44, 42, 43, 57]. In June 2003 it was shown by Tao, [56] that the conjecture in false in dimension 5 and higher. We will not discuss those articles, but concentrate on the important paper [57] by Wang, which also made the first serious attempt at studying $(\mathcal{D},\mathcal{T})$-wavelet sets when $\mathcal{D}$ is not even a subgroup of $\mathrm{GL}(n,\mathbb{R})$, and $\mathcal{T}$ is not a lattice. We need two more definitions before we state some of Wang's results. Let $a \in \mathrm{GL}(n,\mathbb{R})$. A set $\mathcal{D} \subseteq \mathrm{GL}(n,\mathbb{R})$ is said to be a invariant if $\mathcal{D}a = \mathcal{D}$. The multiplicative tiling set $\mathcal{D}$ is said to satisfy the *interior condition* if there exists a multiplicative $\mathcal{D}$-tile Ω such that $\Omega^o \neq \emptyset$. Similarly the spectrum $\mathcal{T} \subset \mathbb{R}^n$ satisfies the *interior condition* if there exists a measurable set $\Omega \subset \mathbb{R}^n$ such that $\Omega^o \neq \emptyset$ and $(\Omega,\mathcal{T})$ is a spectral pair. With these definitions we can state two of Wang's main results:

Theorem 1.10 (Wang,[57])**.** *Let $\mathcal{D} \subset \mathrm{GL}(n,\mathbb{R})$ and $\mathcal{T} \subset \mathbb{R}^n$. Let $\Omega \subset \mathbb{R}^n$ be measurable, with positive and finite measure. If Ω is a multiplicative $\mathcal{D}^T$-tile and $(\Omega,\mathcal{T})$ is a spectral pair, then Ω is a $(\mathcal{D},\mathcal{T})$-wavelet set. Conversely, if Ω is a $(\mathcal{D},\mathcal{T})$-wavelet set and $0 \in \mathcal{T}$, then Ω is a multiplicative $\mathcal{D}^T$-tile and $(\Omega,\mathcal{T})$ is a spectral pair.*

Theorem 1.11 (Wang,[57])**.** *Let $\mathcal{D} \subset \mathrm{GL}(n,\mathbb{R})$ such that $\mathcal{D}^T := \{d^T \mid d \in \mathcal{D}\}$ is a bounded multiplicative tiling set, and let $\mathcal{T} \subset \mathbb{R}^n$ be a spectrum, with both $\mathcal{D}^T$ and $\mathcal{T}$ satisfying the interior condition. Suppose that $\mathcal{D}^T$ is a-invariant for some expanding matrix a and $\mathcal{T} - \mathcal{T} \subset \mathcal{L}$ for some full rank lattice $\mathcal{L}$ of $\mathbb{R}^n$. Then, there exists a wavelet set Ω with respect to $\mathcal{D}$ and $\mathcal{T}$.*

In his paper, Wang states "The assumption that $\mathcal{D}^T$ ... have the interior condition is most likely unnecessary. All known examples of multiplicative tiling sets admit a tile having nonempty interior." In this section, we will in fact show that the assumption that $\mathcal{D}^T$ satisfies the interior condition is indeed unnecessary, but not by proving that every multiplicative tiling set admits a tile having nonempty interior. Instead, we will use a Lebesgue density argument as in [15, 18]. Moreover, the assumption of multiplicative tiling sets having prototiles that are bounded and bounded away from the origin is not a "wavelet" assumption, but rather it is motivated from the point of view of tiling questions and the relation between translation and dilation tilings of the line. From the point of view of wavelets, by Theorem 1.10, one does not always wish to restrict to bounded multiplicative tiling sets. There are, however, some benefits of obtaining wavelet sets that are bounded and bounded away from the origin - especially if they also satisfy some additional properties. For example, if the sets are the finite union of intervals, one can use these wavelets to show that theorems about the poor decay of wavelets in $L^2(\mathbb{R}^n)$ for "bad" dilations are optimal. Along these lines, Bownik [12] showed that if a is irrational and $\psi_1, \ldots, \psi_L$ is an $(a,\mathbb{Z})$-multiwavelet, then there is an i such that for each $\delta > 0$, $\limsup_{|x|\to\infty} |\psi_i|\,|x|^{1+\delta} = \infty$. He also showed that this result is sharp by finding wavelet sets for each of these dilations that are the union of at most three intervals. Another possibility is to use wavelet sets that are the finite union of intervals (and satisfy several extra conditions) as a start point for the smoothing techniques in [16, 40]. However, these two advantages come from having wavelet sets that are not only bounded and bounded away from the origin, but also the finite union of intervals. In the construction considered in [57], it is not clear at all whether the end wavelet sets can be chosen to be the finite union of nice sets. In fact, the construction by Benedetto and Leon was used originally exactly to construct fractal-like wavelet sets.

Since the general question of existence of wavelet sets is phrased not in terms of sets bounded and bounded away from the origin, but arbitrary measurable sets, we will also show that the assumption that there exist a multiplicative tiling set that is bounded and bounded away from the origin is unnecessary. This will be done by showing that whenever there is a set that tiles $\mathbb{R}^n$ by $\mathcal{D}$ dilations, where $\mathcal{D}$ is invariant under an expansive matrix, then there exists a bounded multiplicative tiling set for $\mathcal{D}$.

We begin with some easy observations that were also in [57]. We say that sets U and V in $\mathbb{R}^n$ are a-dilation equivalent if there is a partition $\{U_k \mid k \in \mathbb{Z}\}$ of U such that $\{a^k U_k \mid k \in \mathbb{Z}\}$ is a partition of V.

Lemma 1.12. *Let $\mathcal{D} \subset \mathrm{GL}(n,\mathbb{R})$ be invariant under an invertible matrix a. If Ω is a multiplicative $\mathcal{D}$-tile and Ω_0 is a-dilation equivalent to Ω, then Ω_0 is a multiplicative $\mathcal{D}$-tile.*

Proof. Let S_k be a partition of Ω such that $\Omega_0 \bigcup_{k\in\mathbb{Z}} a^k S_k$. Then,

$$\begin{aligned}\bigcup_{d\in\mathcal{D}} d\Omega_0 &= \bigcup_{d\in\mathcal{D}}\bigcup_{j\in\mathbb{Z}} da^j S_j\\ &= \bigcup_{j\in\mathbb{Z}}\bigcup_{d\in\mathcal{D}} da^j S_j\\ &= \bigcup_{j\in\mathbb{Z}}\bigcup_{d\in\mathcal{D}} dS_j\mathbb{R}^n.\end{aligned}$$

Similarly, one shows that $|d_1\Omega_0 \cap d_2\Omega_0| = 0$ for all $d_1 \neq d_2$ in $\mathcal{D}$. ☐

Lemma 1.13. *Let a be an expansive matrix and Ω_0, Ω_1 be such that for $i = 1, 2$ we have $\left|a^j\Omega_i \cap a^k\Omega_i\right| = 0$ for all $j \neq k$. Then, Ω_0 is a equivalent to Ω_1 if and only if $\cup_{j\in\mathbb{Z}} a^j\Omega_0 = \cup_{j\in\mathbb{Z}} a^j\Omega_1$ a.e.*

Proposition 1.14. *Let $\mathcal{D} \subset \mathrm{GL}(n,\mathbb{R})$ be invariant under an expansive matrix a. If $\mathcal{D}$ is a multiplicative tiling set, then there is a set Ω_0 bounded and bounded away from the origin such that $\{d\Omega_0 \mid d \in \mathcal{D}\}$ tiles $\mathbb{R}^n$. In particular, $\mathcal{D}$ is a bounded multiplicative tiling set.*

Proof. It is widely known that a is expansive if and only if there is an ellipsoid $\mathcal{E}$ such that $\overline{\mathcal{E}} \subset a\mathcal{E}^\circ$. In this case, it is easy to check that $\Omega_1 = a\mathcal{E} \setminus \mathcal{E}$ is a bounded multiplicative tiling set for $\{a^j \mid j \in \mathbb{Z}\}$; that is, Ω_1 is bounded and bounded away from the origin, and $\{a^j\Omega_1 \mid j \in \mathbb{Z}\}$ tiles $\mathbb{R}^n$. Let $S_j = a^j\Omega_1 \cap \Omega$, and $\Omega_0 = \bigcup_{j\in\mathbb{Z}} a^{-j} S_j$. It is clear that $\Omega_0 \subset \Omega_1$, so it is bounded and bounded away from the origin. Moreover, since $\{a^j\Omega_1 \mid j \in \mathbb{Z}\}$ is a tiling of $\mathbb{R}^n$, it follows that $\{S_j \mid j \in \mathbb{Z}\}$ is a partition of Ω; hence, Ω_0 is a-dilation equivalent to Ω. Therefore, by lemma 1.14, Ω_0 is a multiplicative tiling set. ☐

Next, we turn to showing that the assumption of a multiplicative tile with non-empty interior is unnecessary. We have (combining Lemma 2 and Theorem 1 of 1.15):

Theorem 1.15 ([15])**.** *Let M be a measurable subset of $\mathbb{R}^n$ with positive measure satisfying $aM = M$ for some expansive matrix a. Then, there exists a set $E \subset M$ such that $\{E + k \mid k \in \mathbb{Z}^n\}$ tiles $\mathbb{R}^n$ and $\{a^jE \mid j \in \mathbb{Z}\}$ tiles M.*

Suppose that we are considering classes of $(\mathcal{D},\mathcal{T})$-wavelets, where $\mathcal{D} = \{a^j \mid j \in \mathbb{Z}\}$ and $\mathcal{T}$ is a full rank lattice. It is a general principle that one can either assume that a is in (real) Jordan form, in which case one must deal with arbitrary full rank lattices, or one can assume that the lattice $\mathcal{T} = \mathbb{Z}^n$, in which case one needs to consider all matrices of the form bab^{-1}. In particular, if one is working with expansive matrices, it is almost always permissible to restrict attention to translations by $\mathbb{Z}^n$. While this is clear to experts in the field, it is likely that researchers new to this field are not aware that the above theorem is really a theorem about arbitrary full rank lattices.

Indeed, let M be a measurable subset of $\mathbb{R}^n$ with positive measure satisfying $aM = M$, for some expansive matrix a. Let $\mathcal{L}$ be a full rank lattice in $\mathbb{R}^n$. Then, there is an invertible matrix b such that $b\mathcal{L}\mathbb{Z}^n$. The set bM is bab^{-1} invariant, and bab^{-1} is an expansive matrix, so there is a set F such that $\{F + k \mid k \in \mathbb{Z}^n\}$ tiles $\mathbb{R}^n$ and $\{ba^jb^{-1}F \mid j \in \mathbb{Z}\}$ tiles bM. We claim that for $E = b^{-1}F$, $\{E + k \mid k \in \mathcal{L}\}$ tiles $\mathbb{R}^n$ and $\{a^jE \mid j \in \mathbb{Z}\}$ tiles M. Indeed,

$$\begin{aligned}\bigcup_{k\in\mathcal{L}} E + k &= \bigcup_{k\in\mathcal{L}} b^{-1}F + k\\ &= \bigcup_{k\in\mathcal{L}} b^{-1}(F + bk)\\ &= \bigcup_{k\in\mathbb{Z}^n} b^{-1}(F + k)\end{aligned}$$

$$
\begin{aligned}
&= b^{-1}\big(\bigcup_{k\in\mathbb{Z}^n}(F+k)\big) \\
&= \mathbb{R}^n\,.
\end{aligned}
$$

One can similarly show the disjointness of translates by $\mathcal{L}$. To see that $\{a^jE \mid j\in\mathbb{Z}\}$ tiles M, note that

$$
\begin{aligned}
\bigcup_{j\in\mathbb{Z}} a^jE &= \bigcup_{j\in\mathbb{Z}} a^jb^{-1}F \\
&= b^{-1}\big(\bigcup_{j\in\mathbb{Z}} ba^jb^{-1}F\big) \\
&= b^{-1}bM = M.
\end{aligned}
$$

Again, disjointness of the dilates is immediate. Thus, we have proved the following theorem, that seems to be well known:

Theorem 1.16 ([15])**.** *Let M be a measurable subset of $\mathbb{R}^n$ with positive measure satisfying $aM=M$ for some expansive matrix a, and let $\mathcal{T}$ be a full rank lattice in $\mathbb{R}^n$. Then, there exists a set $E\subset M$ such that $\{E+k \mid k\in\mathcal{T}\}$ tiles $\mathbb{R}^n$ and $\{a^jE \mid j\in\mathbb{Z}\}$ tiles M.*

Theorem 1.16 can be used to give an easy proof of Theorem 1.11 removing three of the assumptions, but adding the assumption that the translation set is a full rank lattice.

Theorem 1.17. *Let $\mathcal{D}\subset \mathrm{GL}(n,\mathbb{R})$ be such that $\mathcal{D}^T$ is a multiplicative tiling set. Suppose also that $\mathcal{D}^T$ is a invariant for some expansive matrix a. Let $\mathcal{T}\subset\mathbb{R}^n$ be a full rank lattice. Then, there exists a $(\mathcal{D},\mathcal{T})$-wavelet set E.*

Proof. By assumption, there exists a set Ω such that $\mathcal{D}^T(\Omega)$ is a tiling of $\mathbb{R}^n$. Consider the set $M=\bigcup_{j\in\mathbb{Z}} a^j\Omega$. The set M is clearly a invariant, and $\{a^j(\Omega) \mid j\in\mathbb{Z}\}$ is a measurable partition of M so by 1.16, there exists a set E such that $\{a^j(E) \mid j\in\mathbb{Z}\}$ tiles M and $\{E+k \mid k\in\mathcal{L}^*\}$ tiles $\mathbb{R}^n$. By Lemmas 1.12 and 1.13, since E is a-equivalent to Ω, $\{d^TE \mid d\in\mathcal{D}\}$ tiles $\mathbb{R}^n$. That is, E is a $(\mathcal{D},\mathcal{T})$-wavelet set, as desired. ☐

We have exhibited above the essential nature of the argument in [57]. That is, what is desired is a general criterion for the following question:

Question 5. Given an expansive matrix a, a full rank lattice $\mathcal{L}$ and two sets Ω_1 and Ω_2, when does there exist a set Ω that is a-equivalent to Ω_1 and $\mathcal{L}$ equivalent to Ω_2?

In the above case, we were forced to restrict to the case that Ω_2 is a fundamental region for the full rank lattice $\mathcal{L}$, since that is what was shown in [15]. As a final generalization in this section, we show that what is really necessary is that Ω_2 contain a neighborhood of the origin. The reader should compare the theorem below with the statement and proofs of the theorems in [17] and [18].

Theorem 1.18. *Let a be an expansive matrix and $\Omega_1\subset\mathbb{R}^n$ a set of positive measure such that $|a^j\Omega_1\cap a^k\Omega_1|=0$ whenever $j\neq k$. Let $M=\bigcup_{j\in\mathbb{Z}} a^j\Omega_1$. Let $\mathcal{L}\subset\mathbb{R}^n$ be a full rank lattice and $\Omega_2\subset\mathbb{R}^n$ such that $|\Omega_2+k_1\cap\Omega_2+k_2|=0$ for $k_1\neq k_2\in\mathcal{L}$ and such that there exists $\epsilon>0$ such that $M\cap B_\epsilon(0)\subset\Omega_2\cap B_\epsilon(0)$. Then, there exists a set Ω such that Ω is a equivalent to Ω_1 and $\mathcal{L}$ equivalent to Ω_2.*

Before proving Theorem 1.18, we state and prove its main corollary, which is Theorem 2.1 of [57] with all but one technical assumption removed.

Corollary 1.19. *Let $\mathcal{D}\subset\mathrm{GL}(n,\mathbb{R})$ be such that $\mathcal{D}^T$ is a multiplicative tiling set. Let $\mathcal{T}$ be a spectrum with interior such that there exists a full rank lattice such that $\mathcal{T}-\mathcal{T}\subset\mathcal{L}$. Then, if $\mathcal{D}^T$ is a-invariant for some expansive matrix a, there exists a $(\mathcal{D},\mathcal{T})$-wavelet set.*

Proof. Since translations of spectral sets are again spectral sets, we may assume without loss of generality that Ω_2 contains 0 as an interior point. By Lemma 3.1 of [57], $(\Omega_2+k_1)\cap(\Omega_2+k_2)$ has measure 0 whenever $k_1 \neq k_2 \in \mathcal{L}^*$. So, by 1.18, there is a set Ω that is a equivalent to Ω_1 and $\mathcal{L}^*$ equivalent to Ω_2. By Lemma 1.12, $\{d^T\Omega \mid d \in \mathcal{D}\}$ tiles $\mathbb{R}^n$, and by Lemma 3.2 in [57], $(\Omega, \mathcal{T})$ is a spectral set. Therefore, Ω is a $(\mathcal{D}, \mathcal{T})$-wavelet set. □

We turn now to proving Theorem 1.18. We begin by noting that arguing as in the proof of Theorem 1.16, one can restrict to the case $\mathcal{L} = \mathbb{Z}^n$. Next, we need to extract the following lemma from the proof of Corollary 1 in [15], then we will follow very closely the proof in [18].

Lemma 1.20. *Let a be an expansive matrix in $GL(n,\mathbb{R})$. Let F_0 be a set of positive measure such that $|a^jF_0 \cap a^kF_0| = 0$ whenever $j \neq k$. Let $E = [-1/2, 1/2]^n$. Then, for every $\epsilon > 0$, there exists $k_0 \in \mathbb{Z}$ and $\ell_0 \in \mathbb{Z}^n$ such that $|a^{k_0}F_0 \cap (E + \ell_0)| > 1 - \epsilon$.*

The proof of Lemma 1.20 is a clever use of a Lebesgue density argument, which we will not repeat here.

Proof of Theorem 1.18. First, note that as in the case of Theorem 1.16, Lemma 1.20 is really a lemma about arbitrary full rank lattices $\mathcal{L}$. Moreover, one can replace $E = [-1/2, 1/2]^n$ by any subset E of a fundamental region of $\mathcal{L}$ to get the following formally stronger lemma.

Lemma 1.21. *Let a be an expansive matrix in $GL(n,\mathbb{R})$. Let F_0 be a set of positive measure such that $|a^jF_0 \cap a^kF_0| = 0$ whenever $j \neq k$. Let $\mathcal{L} \subset \mathbb{R}^n$ be a full rank lattice with fundamental region Ω. Then, for every set $E \subset \Omega$ and every $\epsilon > 0$, there exists $k_0 \in \mathbb{Z}$ and $\ell_0 \in \mathcal{L}$ such that $|a^{k_0}F_0 \cap (E + \ell_0)| \geq (1-\epsilon)|E|$.*

Turning to the proof of 1.18, note that by Theorem 1.14, we may assume without loss of generality that Ω_1 is bounded and bounded away from the origin. We may also assume that Ω_2 is contained in a convex, centrally symmetric fundamental region of $\mathcal{L}$. The rest of the proof follows very closely the proof of Theorem 3.7 in [18], with Lemma 1.21 playing the role of Proposition 3.5 in [18]. We will construct a family $\{G_{ij} \mid i \in \mathbb{N}, j \in \{1,2\}\}$ of measurable sets whose a-dilates form a measurable partition of Ω_1 and whose translates by vectors in $\mathcal{L}$ form a measurable partition of Ω_2. Then

$$\Omega := \bigcup G_{ij} \tag{1.1.1}$$

is the set desired in Theorem 1.18. Since the steps are so similar to [18], we will give the first step of the inductive definition, and the properties needed for induction. Details are the same as in [18].

Let $\{\alpha_i\}$ and $\{\beta_i\}$ be sequences of positive constants decreasing to 0 and such that $\alpha_1 < \epsilon$ chosen so that $B_{\alpha_1}(0) \cap M \subset B_{\alpha_1}(0) \cap \Omega_2$. Let $\tilde{E}_{11} = \Omega_2 \setminus B_{\alpha_1}(0)$. Then, $|(\tilde{E}_{11})| > 0$. Let $\tilde{F}_{11}$ be a measurable subset of Ω_1 with measure strictly less than $|\Omega_1|$. By Lemma 1.21, there exists $k_1 \in \mathbb{N}$ and $\ell_1 \in \mathcal{L}$ such that

$$|a^{k_1}\tilde{F}_{11} \cap (\tilde{E}_{11} - \ell_1)| \geq \frac{1}{2}|\tilde{E}_{11}| \tag{1.1.2}$$

Let $G_{11} := a^{k_1}\tilde{F}_{11} \cap (\tilde{E}_{11} - \ell_1)$, let $E_{11} := G_{11} + \ell_1$, and let

$$F_{11} := \tilde{F}_{11} \cap a^{-k_1}(E_{11} - \ell_1)a^{-k_1}G_{11} \tag{1.1.3}$$

Then $F_{11} \subset \tilde{F}_{11} \subset \Omega_1$ and $|\Omega_1 \setminus \tilde{F}_{11}| \geq |\Omega_1 \setminus F_{11}| > 0$. Also, $E_{11} \subset \tilde{E}_{11}$, and

$$|E_{11}| = |G_{11}| \geq \frac{1}{2}|\tilde{E}_{11}| \tag{1.1.4}$$

Also, $G_{11} = a^{k_1}F_{11}$. Now choose $F_{12} \subset \Omega_1$, disjoint from F_{11}, such that $\Omega_1 \setminus (F_{11} \bigcup F_{12})$ has positive measure less than β_1. Choose m_1 such that $a^{-m_1}F_{12}$ is contained in $N_1 = B_{\alpha_1/2}(0)$ and is disjoint from G_{11}. (This is possible since G_{11} is bounded away from 0.) Set

$$G_{12} := E_{12} := a^{-m_1}F_{12}. \tag{1.1.5}$$

The first step is complete.

Proceed inductively, obtaining disjoint families of positive measure $\{E_{ij}\}$ in Ω_2, $\{F_{ij}\}$ in Ω_1 and $\{G_{ij}\}$ such that for $i = 1, 2, \dots$ and $j = 1, 2$ we have

1. $G_{i1} + \ell_i = E_{i1}$;
2. $G_{i2} = E_{i2}$;
3. $a^{-k_1} G_{i1} = F_{i1}$;
4. $a^{m_i} G_{i2} = F_{i2}$;
5. $|\Omega_1 \setminus (F_{11} \bigcup F_{12} \bigcup \cdots \bigcup F_{i1} \bigcup F_{i2})| < \beta_i$, and
6. $|E_{i1}| \geq \frac{1}{2} |\Omega_2 \setminus (E_{11} \bigcup E_{12} \bigcup \cdots \bigcup E_{i-1,1} \bigcup E_{i-1,2})) - \frac{1}{2}|N_i|$, where N_i is a ball centered at 0 with radius less than α_i.

Since $\beta_i \to 0$, item 5 implies that $F \setminus (\bigcup F_{ij})$ is a null set, and since $\alpha_i \to 0$, item 6 implies that $(E \setminus (\bigcup E_{ij}))$ is a null set. Let

$$F = \bigcup \{G_{ij} \mid i = 1, 2, \dots, j = 1, 2\} \tag{1.1.6}$$

then, G is congruent to Ω_2 by items 1 and 2, and the a dilates of G form a partition of M, as desired. □

For sets $\mathcal{D} \subset \mathrm{GL}(n, \mathbb{R})$ which are invariant under an expansive matrix, Corollary 1.19 is nice in that it reduces the question of existence of wavelet sets to the the question of existence of tiling sets for dilations and translations separately. It is still in some sense unsatisfactory, because it relies on the existence of objects external to the sets $(\mathcal{D}, \mathcal{T})$ under consideration. From the point of view of characterizing sets $(\mathcal{D}, \mathcal{T})$ for which wavelet sets exist, something more is needed. We will present in section 4 some progress on this question when $\mathcal{D}$ is a countable subgroup of $GL(n, \mathbb{R})$.

2. Admissible groups and frames

We will now turn to the applications of those results to frames and wavelets in $\mathbb{R}^n$. But first we recall some results about the continuous wavelet transform.

Recall that translations and dilations on the real line form the so-called $(ax + b)$-group. Assume in general that we have a group G acting on a topological space. Assume that μ is a Radon measure on X and that the measure $g \cdot \mu : E \mapsto \mu(g^{-1} \cdot E)$ is absolutely continuous with respect to μ. Then μ is *quasi-invariant*, i.e, there exists a measurable function $j : G \times X \to \mathbb{R}^+$ such that

$$\int_X f(g \cdot x)\, d\mu(x) = \int_X j(g, x) f(x)\, d\mu(x)$$

for all $f \in L^1(X)$. Then, we can define a unitary representation of G on $L^2(X)$ by

$$[\pi(g)f](x) = j(g^{-1}, x)^{-1/2} f(g^{-1}x)\,.$$

For a fixed $\psi \in L^2(G)$ define the transform $W_\psi : L^2(X) \to C(G)$ by

$$W_\psi(f)(g) := (f, \pi(g)\psi)$$

and notice that W_ψ intertwines the representation π and the *left regular representation*, i.e.,

$$\begin{aligned} W_\psi(\pi(g)f)(x) &= (\pi(g)f, \pi(x)\psi) \\ &= (f, \pi(g^{-1} \cdot x)\psi) \\ &= W_\psi(f)(g^{-1} \cdot x)\,. \end{aligned}$$

Notice that $W_\psi(f)(g) \leq \|f\|_2 \|\psi\|_2$, and hence $W_\psi(f)$ is bounded. One of the important question now is:

Question 6. Given the group G acting on X find a discrete subset $\mathcal{D} \subseteq G$ and a function ψ such that $\{\pi(g^{-1})\psi \mid g \in D\}$ is a frame for $L^2(X)$.

For a general discussion we refer to the fundamental work of Feichtinger and Gröchenig [23, 24].

Definition 2.1. Let $\mathcal{H}$ be a separable Hilbert space, and let $\mathbb{J}$ be a finite our countable infinite index set. A sequence $\{f_j\}_{j\in\mathbb{J}}$ in $\mathcal{H}$ is called a *frame* if there exists constants $0 < A \leq B < \infty$ such that for all $x \in \mathcal{H}$ we have.

$$A\|x\|^2 \leq \sum_{j\in\mathbb{J}} |(x, f_j)|^2 \leq B\|x\|^2 .$$

$\{f_j\}_{j\in\mathbb{J}}$ is a *tight frame* if we can choose $A = B$ and a *normalized tight frame* or *Parseval frame* if we can choose $A = B = 1$.

Example 2.2. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{K} \subset \mathcal{H}$ a closed subspace. Assume that $\{u_n\}$ is a orthonormal basis of $\mathcal{H}$. Let $\mathrm{pr} : \mathcal{H} \to \mathcal{K}$ be the orthogonal projection. Define $f_j = \mathrm{pr}(u_j)$. Then $\{f_j\}$ is a Parseval frame for $\mathcal{K}$. In fact it is easy to see that every Parseval frame can be constructed in this way. In particular we can apply this to the situation where $(\Omega, \mathcal{T})$ is a spectral pair and $M \subset \Omega$ is measurable with $|M| > 0$. Then $\{|\Omega|^{-1/2}e_\lambda\}_{\lambda\in\mathcal{T}}$ is an orthonormal basis for $L^2(\Omega)$ and hence $\{f_\lambda := |\Omega|^{-1/2}e_\lambda|_M\}_{\lambda\in\mathcal{T}}$ is a Parseval frame for $L^2(M)$.

Example 2.3. For the $(ax + b)$-group $j(a, x) = |a|^{-1}$ is independent of x and we get:

$$\begin{aligned} W_\psi(f)(a, b) &= (f, \pi(a, b)\psi) \\ &= (f, T_b D_a \psi) \\ &= |a|^{-1/2} \int_{\mathbb{R}} f(x)\overline{\psi((x-b)/a)}\, dx . \end{aligned}$$

Here $T_b : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ stands for the unitary isomorphism corresponding to translation $T_b f(x) = f(x - b)$ and $D_a : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is the unitary map corresponding to dilation $D_a f(x) = |a|^{-1/2} f(x/a)$, $a \neq 0$.

The discrete wavelet transform is obtained by sampling the wavelet transform $W_\psi(f)$ at points gotten by replacing the full $(ax+b)$-group by a discrete subset generated by translation by integers and dilations of the form $a = 2^n$:

$$\begin{aligned} W^d_\psi(f)(2^{-n}, -2^{-n}m) &= (f \mid \pi((2^n, m)^{-1})\psi) \\ &= 2^{n/2} \int_{\mathbb{R}} f(x)\overline{\psi(2^n x + m)}\, dx \\ &= (f, D_{2^{-n}} T_{-m}\psi) . \end{aligned}$$

Hence, the corresponding frame is

$$\{\pi((2^n, m)^{-1})\psi \mid n, m \in \mathbb{Z}\} = \{D_{2^n} T_m \psi \mid n, m \in \mathbb{Z}\} . \tag{2.2.1}$$

The inverse refers here to the inverse in the $(ax + b)$-group.

As in the last example, it is well known, that the short time Fourier transform, and several other well known integral transforms have a common explanation in this way in the language of representation theory. This observation is the basis for the generalization of the continuous wavelet transform to higher dimensions and more general settings, and was already made by A. Grossmann, J. Morlet, and T. Paul in 1985, see [34, 35]. In [34] the connection to square integrable representations and the relation to the fundamental paper of M. Duflo and C. C. Moore [20] was already pointed out. Several natural questions arise now, in particular to describe the image of the transform W_ψ and how that depends on ψ. But we will not go into that here, but refer to [2, 3, 6, 8, 19, 21, 24, 25, 28, 30, 31, 32, 33, 34, 35, 39, 46, 51, 59] for discussion. Here, we will concentrate on the connection to frames, wavelets and wavelet sets.

Denote by $\mathrm{Aff}(\mathbb{R}^n)$ the group of invertible affine linear transformations on $\mathbb{R}^n$. Thus $\mathrm{Aff}(\mathbb{R}^n)$ consists of pairs (x, h) with $h \in \mathrm{GL}(n, \mathbb{R})$ and $x \in \mathbb{R}^n$. The action of $(x, h) \in \mathrm{Aff}(\mathbb{R}^n)$ on $\mathbb{R}^n$ is given by

$$(x, h)(v) = h(v) + x .$$

The product of group elements is the composition of maps. Thus

$$(x,a)(y,b) = (a(y)+x, ab)$$

the identity element is $e = (0, \mathrm{id})$ and the inverse of $(x,a) \in \mathrm{Aff}(\mathbb{R}^n)$ is given by

$$(x,a)^{-1} = (-a^{-1}(x), a^{-1})\,. \tag{2.2.2}$$

Thus $\mathrm{Aff}(\mathbb{R}^n)$ is the semidirect product of the abelian group $\mathbb{R}^n$ and the group $\mathrm{GL}(n,\mathbb{R})$; $\mathrm{Aff}(\mathbb{R}^n) = \mathbb{R}^n \times_s \mathrm{GL}(n,\mathbb{R})$. Let $H \subseteq \mathrm{GL}(n,\mathbb{R})$ be a closed subgroup. (In fact it is not necessary to assume that H is closed, but for simplicity we will assume that.) Define

$$\mathbb{R}^n \times_s H := \{(x,a) \in \mathrm{Aff}(\mathbb{R}^n) \mid a \in H,\ x \in \mathbb{R}^n\}.$$

Then $H \times_s \mathbb{R}^n$ is a closed subgroup of $\mathrm{Aff}(\mathbb{R}^n)$, and hence a Lie group. *From now on H will always denote a closed subgroup of* $\mathrm{GL}(n,\mathbb{R})$.

Define a unitary representation of $\mathbb{R}^n \times_s H$ on $L^2(\mathbb{R}^n)$ by

$$[\pi(x,a)f](v) := |\det(a)|^{-1/2} f((x,a)^{-1}(v)) = |\det(a)|^{-1/2} f(a^{-1}(v-x))\,. \tag{2.2.3}$$

Write $\psi_{x,a}$ for $\pi(x,a)\psi$. Because of the role played by the Fourier transform, we will also need another action of H on $\mathbb{R}^n$ given by $a \cdot \omega := (a^{-1})^T(\omega)$. We denote by $\hat{\pi}(x,a)$ the unitary action on $L^2(\mathbb{R}^n)$ given by

$$\hat{\pi}(x,a)f(v) = \sqrt{|\det(a)|}\, e^{-2\pi i (x|v)} f(a^{-1}\cdot v) = \sqrt{|\det(a)|}\, e^{-2\pi i (x|v)} f(a^T(v))\,. \tag{2.2.4}$$

Remark 2.4. Some authors use the semidirect product $H \times_s \mathbb{R}^n$ instead of $R^n \times_s H$. Thus first the translation and then the linear map is applied, i.e., $(a,x)(v) = a(v+x)$. In this notation the product becomes $(a,x)(b,y) = (ab, ab(y) + a(x))$, the inverse of (a,x) is $(a,x)^{-1} = (a^{-1}, -ax)$, and the wavelet representation is

$$\widetilde{\pi}(a,x)f(v) = |\det a|^{-1/2} f(a^{-1}v - x)\,. \tag{2.2.5}$$

Furthermore, instead of using the transposed action, we could just as well represent $\mathbb{R}^n$ in the frequency picture as row vectors and use the right action.

The Fourier transform intertwines the representations π and $\hat{\pi}$ [21], Lemma 3.1, i.e.,

$$\widehat{\pi(x,a)f}(\omega) = \hat{\pi}(x,a)\hat{f}(\omega)\,, \quad f \in L^2(\mathbb{R}^n)\,. \tag{2.2.6}$$

Denote by $d\mu_H$ a left invariant measure on H. A left invariant measure on G is then given by $d\mu_G(x,a) = |\det(a)|^{-1} d\mu_H(a)dx$. Let $f, \psi \in L^2(\mathbb{R}^n)$. A simple calculation shows that

$$\int_G |(f \mid \pi(x,a)\psi)|^2\, d\mu_G(x,a) = \int_{\mathbb{R}^n} |\hat{f}(\omega)|^2 \int_H |\hat{\psi}(a^{-1}\cdot\omega)|^2\, d\mu_H(a) d\omega \tag{2.2.7}$$

$$= \int_{\mathbb{R}^n} |\hat{f}(\omega)|^2 \int_H |\hat{\psi}(a^T(\omega))|^2\, d\mu_H(a) d\omega \tag{2.2.8}$$

There are several ways to read this. First let $M \subseteq \mathbb{R}^n$ be measurable and invariant under the action $H \times \mathbb{R}^n \ni (h,v) \mapsto h \cdot v := (h^{-1})^T(v) \in \mathbb{R}^n$. Then we denote by $L^2_M(\mathbb{R}^n)$ the space of function $f \in L^2(\mathbb{R}^n)$ such that $\hat{f}(\xi) = 0$ for almost all $\xi \notin M$. Notice that $L^2_M(\mathbb{R}^n)$ is a closed invariant subspace, and that the orthogonal projection onto $L^2_M(\mathbb{R}^n)$ is given by $f \mapsto (\hat{f}\chi_M)^\vee$. The first result is now:

Theorem 2.5. *Let $M \subseteq \mathbb{R}^n$ be measurable of positive measure, and invariant under the action $(a,v) \mapsto a \cdot v$. Then the wavelet transform*

$$W_\psi : f \mapsto (f \mid \pi(x,a)\psi)_{L^2(\mathbb{R}^n)} = |\det a|^{-1/2} \int f(y)\overline{\psi(a^{-1}(y-x))}\, dy$$

is a partial isometry $W_\psi : L^2_M(\mathbb{R}^n) \to L^2(G)$ *if and only if*

$$\Delta_\psi(\omega) := \int_H |\hat{\psi}(a^T\omega)|^2 \, d\mu_H(a) = 1 \tag{2.2.9}$$

for almost all $\omega \in M$.

Motivated by [46, 59] we define

Definition 2.6 (Laugesen, Weaver, Weiss, and Wilson)**.** Let $M \subseteq \mathbb{R}^n$, be measurable, invariant, and $|M| > 0$. Let $\psi \in L^2(\mathbb{R}^n)$ then ψ is said to be *a (normalized) admissible* (H, M)*-wavelet* if for almost all $\omega \in M$ we have

$$\int_H |\hat{\psi}(a^T\omega)|^2 \, d\mu_H(a) = 1 \ .$$

We say that the pair (H, M) is *admissible* if a (H, M)-admissible wavelet ψ exists. If $M = \mathbb{R}^n$ then we say that H is *admissible* and that ψ is a (normalized) *wavelet* function.

Assume that $\psi \in L^2_M(\mathbb{R}^n)$ is a normalized admissible wavelet. Then

$$G \ni (x, a) \mapsto F(x, a) := W_\psi f(x, a)\psi_{x,a} \in L^2_M(\mathbb{R}^n)$$

is well defined and if $g \in L^2_M(\mathbb{R}^n)$ then

$$\begin{aligned}
\int_G (F(x, a) \mid g)_{L^2(\mathbb{R}^n)} \, d\mu_G(x, a) &= \int_G W_\psi f(x, a) \left(\int_{\mathbb{R}^n} \psi_{x,a}(y)\overline{g(y)} \, dy \right) \, d\mu_G(x, a) \\
&= \int_G W_\psi f(x, a)\overline{W_\psi g(x, a)} \, d\mu_G(x, a) \\
&= (W_\psi f \mid W_\psi g)_{L^2(G)} \\
&= (f \mid g)_{L^2(\mathbb{R}^n)} \ .
\end{aligned}$$

Hence

Lemma 2.7. *Assume that* $\psi \in L^2_M(\mathbb{R}^n)$ *satisfies* $\int_H |\hat{\psi}(a^T\omega)|^2 \, d\mu_H(a) = 1$ *for almost all* $\omega \in M$. *Then, as a weak integral,*

$$f = \int W_\psi f(x, a)\psi_{x,a} \, d\mu_G(x, a) \tag{2.2.10}$$

for all $f \in L^2(\mathbb{R}^n)$.

Question 7 (Laugesen, Weaver, Weiss, and Wilson)**.** Give a characterization of admissible subgroups of $\mathrm{GL}(n, \mathbb{R})$.

It is easy to derive one necessary condition for admissibility. For $\omega \in \mathbb{R}^n$ let

$$H^\omega = \{h \in H \mid h \cdot \omega = \omega\} = \left\{h \in H \mid h^T(\omega) = \omega\right\} \tag{2.2.11}$$

be the stabilizer of ω. Then admissibility implies that H is compact for almost all $\omega \in \mathbb{R}^n$. This condition is not sufficient and by now, there is *no* complete characterization of admissible group. The best result is the following, due to Laugesen, Weaver, Weiss, and Wilson [46]:

Theorem 2.8 (Laugesen, Weaver, Weiss, and Wilson, [46])**.** *Let* H *be a closed subgroup of* $\mathrm{GL}(n, \mathbb{R})$. *For* $\omega \in \mathbb{R}^n$ *and* $\epsilon > 0$ *let*

$$H^\omega_\epsilon := \{h \in H \mid \|a \cdot \omega - \omega\| \le \epsilon\}$$

be the ϵ*-stabilizer of* ω. *If either*

1. $G = \mathbb{R}^n \times_s H$ *is non-unimodular, or*
2. $\{\omega \in \mathbb{R}^n \mid H^\omega$ *is non-compact* $\}$ *has positive Lebesgue measure*

holds, then H is not admissible. If both (1) and

(3) $\{\omega \in \mathbb{R}^n \mid H^\omega_\epsilon$ *is non-compact for all* $\epsilon > 0\}$ *has positive Lebesgue measure fail,*

then H is admissible.

Remark 2.9. If M is homogeneous under H^T. Then (H, M) is admissible if and only if H^ω is compact for one - and hence all $-$ $\omega \in M$, c.f. Lemma 3.1.

Various versions of the following discrete version of the wavelet transform are well known in the literature. From now on H will always stand for a closed subgroup of $\mathrm{GL}(n, \mathbb{R})$.

Theorem 2.10. *Let $M \subseteq \mathbb{R}^n$ be measurable with positive measure and such that $\left|\overline{M} \setminus M\right| = 0$. Assume that there exist a countable, discrete subset $\Gamma \subseteq \mathrm{GL}(n, \mathbb{R})$ and a measureable set $\mathbb{F} \subset M$ such that the following holds:*

1. $|M \setminus \bigcup_{\gamma \in \Gamma} \gamma^T \mathbb{F}| = 0$;
2. $\sup_{\gamma \in \Gamma} \#\{\sigma \in \Gamma \mid \left|\sigma^T \mathbb{F} \cap \gamma^T \mathbb{F}\right| \neq 0\} = L < \infty$;
3. *There exists a set $\mathcal{T} \subset \mathbb{R}^n$, such that*
$$\{e_t|_{\mathbb{F}} \mid t \in \mathcal{T}\}$$
is a frame for $L^2(M)$ with frame constants $0 < A \leq B$.

Let $\psi = \chi_{\mathbb{F}}^{\vee}$. Then
$$\{\pi((t, \gamma)^{-1})\psi \mid t \in \mathcal{T},\, \gamma \in \Gamma\}$$
is a frame for $L^2_M(\mathbb{R}^n)$ with frame constants A and LB. In particular if $L = 1$ and $\{e_t|_{\mathbb{F}} \mid t \in \mathcal{T}\}$ is tight frame, then $\{\pi((t, \gamma)^{-1})\psi \mid t \in \mathcal{T},\, \gamma \in \Gamma\}$ is a tight frame, with same frame constant.

Proof. First recall that $(t, \gamma)^{-1} = (-\gamma^{-1}, \gamma^{-1})$ by equation (2.2.2). Hence by (2.2.4) and (2.2.6) it follows that (with $\gamma^{\#} = (\gamma^{-1})^{-1}$:
$$\begin{aligned}
(f, \pi((t, \gamma)^{-1})\psi) &= (\hat{f}, \pi(\widehat{(t, \gamma)^{-1}})\psi) \\
&= |\det(\gamma)|^{-1/2} \int \hat{f}(\lambda) e^{-2\pi(\gamma^{-1} t, \lambda)} \chi_{\mathbb{F}}(\gamma \cdot \lambda)\, d\lambda \\
&= \sqrt{|\det(\gamma)|} \int_{\mathbb{F}} \hat{f}(\gamma^T \lambda) e^{-2\pi(t, \lambda)}\, d\lambda \\
&= ([D_{\gamma^{\#}} \hat{f}]|_{\mathbb{F}}, e_t|_{\mathbb{F}})\,.
\end{aligned}$$
As $\{e_t|_{\mathbb{F}} \mid t \in \mathcal{T}\}$ is a frame for $L^2(\mathbb{F})$ with frame bounds A and B, it follows that
$$A\|[D_{\gamma^{\#}} \hat{f}]|_{\mathbb{F}}\|^2 \leq \sum_{t \in \mathcal{T}} |([D_{\gamma^{\#}} \hat{f}]|_{\mathbb{F}}, e_t)|^2 \leq B\|[D_{\gamma^{\#}} \hat{f}]|_{\mathbb{F}}\|^2$$
or
$$A\|\hat{f}\chi_{\gamma^T \mathbb{F}}\|^2 \leq \sum_{t \in \mathcal{T}} |(f, \pi((t, \gamma)^{-1}))\psi)|^2 \leq B\|\hat{f}\chi_{\gamma^T \mathbb{F}}\|^2\,.$$
But by assumptions (1) and (2), it follows that
$$|\hat{f}|^2 \leq \sum_{\gamma \in \Gamma} |\hat{f}|^2 \chi_{\gamma^T \mathbb{F}} \leq L|\hat{f}|^2$$
and the statement of the theorem follows. □

Few remarks about this theorem are at place here:

Remark 2.11. If $\mathbb{F}$ is bounded and satisfies (1) and (2) then we there are always infinitely many $\mathcal{T}$ satisfying (2). Just, c.f., [8], p. 605, take any parallelpiped

$$R = \{\sum_{j=1}^{n} t_j u_j \mid a_j \leq t_j \leq b_j,\, j = 1, \dots , n\}$$

such that $\mathbb{F} \subseteq R$. Then $u_1, \dots , u_n$ is a basis for $\mathbb{R}^n$. Let $v_1, \dots , v_n$ be the dual basis, and define

$$\mathcal{T} := \{\sum_{j=1}^{n} \frac{k_j}{b_j - a_j} v_j \mid k = (k_1, \dots , k_n) \in \mathbb{Z}^n\} .$$

Then $\{|R|^{-1/2} e_t|_R \mid t \in \mathcal{T}\}$ is an orthonormal basis for $L^2(R)$. By Example 2.2 it follows that $\{|R|^{-1/2} e_t|_{\mathbb{F}} \mid t \in \mathcal{T}\}$ is a tight frame for $L^2(\mathbb{F})$.

Remark 2.12. The function ψ in the above Theorem is non smooth, as its Fourier transform is an indicator function and hence not even continuous. But instead of ψ we can choice a smooth, compactly supported function h such that there exists constants $0 \leq a \leq b$ such that $a \leq |h|_{\mathbb{F}} \leq$ and such that the support of h satisfies condition (2) (possibly with another L). Then a simple modification of the above proof shows that $\{\pi((t, \gamma)^{-1} h^{\vee}) \mid (t, \gamma) \in \mathcal{T} \times \Gamma\}$ is a frame, see [8], Theorem 3.

In the above theorem we did not assume that M is invariant under a subgroup of $\mathrm{GL}(n, \mathbb{R})$ or that Γ has any special structure. But in applications, or for constructing examples, we will later use that M is invariant under a group H and then use the structure of H to construct both Γ and $\mathbb{F}$. The following theorem is the prime tool for that. We will need the following, which is a part of Lemma 3.3 in [23]:

Lemma 2.13 (Feichtinger and Gröchenig)**.** *Let H be a locally compact, Hausdorff, topological group. Let $\Gamma \subset G$. Then the following conditions are equivalent:*

1. *There exists a relatively compact open subset $V \subset H$, $e \in H$, such that $\gamma V \cap \sigma V = \emptyset$ if $\gamma, \sigma \in \Gamma$, $\gamma \neq \sigma$;*
2. *Given any relatively compact subset W in H such that $W^o \neq \emptyset$, then*

$$\sup_{\gamma \in \Gamma} \#\{\sigma \in \Gamma \mid \gamma W \cap \sigma W \neq \emptyset\} < \infty|, .$$

Theorem 2.14. *Assume that M is H-homogeneous under the action $(h, x) \mapsto h \cdot x$, and that (H, M) is admissible. Suppose there exists a relatively compact measurable set $\mathbb{F}_H \subseteq H$ with non-empty interior, and a discrete set $\Gamma \subseteq H$ such that the following holds:*

1. $\mu_H(H \setminus \bigcup_{\gamma \in \Gamma} \gamma \mathbb{F}_H) = 0$*;*
2. $\sup_{\gamma \in \Gamma} \#\{\sigma \in \Gamma \mid \sigma \mathbb{F}_H \cap \gamma \mathbb{F}_H \neq \emptyset\} < \infty$*.*

Let $\omega \in M$ and set $\mathbb{F} = \mathbb{F}_H \cdot \omega$. Then the pair $(\Gamma^{-1}, \mathbb{F})$ satisfies the conditions (1), (2), and (3) in Theorem 2.10.

Proof. As M is homogeneous it follows that $M \simeq H/H^{\omega}$, where $\simeq$ stands for diffeomorphic. Hence $\mathbb{F}$ is measurable with finite measure. Then notice that

$$\begin{aligned} \bigcup_{\gamma \in \Gamma^{-1}} \gamma^T \mathbb{F} &= \bigcup_{\gamma \in \Gamma} \gamma \cdot \mathbb{F}_H \cdot \omega \\ &= \bigcup_{\gamma \in \Gamma} (g \mathbb{F}_H) \cdot \omega \\ &= H \cdot \omega \\ &= M . \end{aligned}$$

By Lemma 2.13 there exists an open, relatively compact set $V \subset H$, with $e \in V$, such that

$$\gamma V \cap \sigma V = \emptyset\,, \qquad \gamma \neq \sigma\,. \tag{2.2.12}$$

As H^ω is compact, it follows that $K := \mathbb{F}_H H^\omega$ is relatively compact. Hence, by (2.2.12) and Lemma 2.13, we get:

$$\sup_{\gamma\in\Gamma} \#\{\sigma\in\Gamma \mid \gamma K \cap \sigma K \neq \emptyset\} =: L < \infty\,. \tag{2.2.13}$$

By Lemma 1.4 in [51] we have

$$\sup_{\gamma\in\Gamma^{-1}} \#\{\sigma\in\Gamma^{-1} \mid \gamma^T\mathbb{F} \cap \sigma^T\mathbb{F} \neq \emptyset\} = \sup_{\gamma\in\Gamma} \#\{\sigma\in\Gamma \mid \gamma K \cap \sigma K \neq \emptyset\} =: L < \infty \tag{2.2.14}$$

and condition (2) in Theorem 2.10. As $\overline{\mathbb{F}_H}$ is compact, it follows that $\overline{\mathbb{F}}$ is compact. In particular $\mathbb{F}$ is bounded and the existence of the set $\mathcal{T}$ in (3) follows by Remark 2.11. □

Corollary 2.15. *Assume that M is homogeneous and that (H, M) is admissible. If there exists a co-compact subgroup $\Gamma \subseteq H$, then there exists a set $\mathbb{F} \subseteq M$ satisfying the conditions in Theorem 2.10*

3. The FO condition and frames

Theorem 2.14 combined with Theorem 2.10 gives us a tool to construct frames in subsets of $\mathbb{R}^n$. The problem becomes to find groups such that $(H, \mathbb{R}^n)$ is admissible and such that we can apply Theorem 2.14. In this section we discuss a special class of such group introduced in [21] and [51]. Those are admissible groups with finitely many open orbits. They groups are closely related to the so-called *prehomogeneous vector spaces of parabolic type* [9]. We say that the pair $(H.M)$, $H \subseteq \mathrm{GL}(n,\mathbb{R})$, $\emptyset \neq M \subseteq \mathbb{R}^n$, satisfies the condition *FO* if M is H^T-invariant, and there exists finitely many H^T-invariant open and disjoint sets $U_1, \ldots, U_k \subseteq M$, such that each U_j is H^T-invariant and homogeneous, and $\left| M \setminus \bigcup_{j=1}^k U_j \right| = 0$. We start with a simple lemma:

Lemma 3.1. *Assume that the pair (H, M) satisfies the condition FO. Then (H, M) is admissible if and only if the stabilizer H^ω is compact for $\omega \in U_j$, $j = 1, \ldots, k$.*

Proof. It is clear, that if (H, M) is admissible, then H^ω is compact for each $\omega \in \cup_j U_j$. For the other direction, fix $\omega_j \in U_j$. For $j = 1, \ldots, k$ let $g_j \in C_c(U_j)$, $g_j \geq 0$, $g \neq 0$. Then the function

$$H \ni a \mapsto g_j(a^T(\omega_j)) \in \mathbb{C}$$

has compact support and $\int_H g_j(a^T(\omega_j))\, d\mu_H(a) > 0$. Let $\omega \in U_j$. Choose $h \in H$ such that $\omega = h^T(\omega_j)$. This is possible because U_j is homogeneous under H^T. Then

$$\begin{aligned} \int_H g_j(a^T(\omega))\, d\mu_H(a) &= \int_H g_j(a^T h^T(\omega_j))\, d\mu_H(a) \\ &= \int_H g_j((ha)^T(\omega_j))\, d\mu_H(a) \\ &= \int_H g_j(a^T(\omega_j))\, d\mu_H(a)\,. \end{aligned}$$

Hence $\Delta_j = \int_H g_j(a^T(\omega))\, d\mu_H(a) > 0$ is independent of $\omega \in U_j$. Define $\varphi : \mathbb{R}^n \to \mathbb{C}$ by

$$\varphi(\omega) := \begin{cases} g_j(\omega)/\Delta_j & \text{if } \omega \in U_j \\ 0 & \text{if } \omega \notin \bigcup_{j=1}^k U_j \end{cases}\,.$$

Then $\varphi \in C_c(\mathbb{R}^n)$, so in particular $\varphi \in L^2(\mathbb{R}^n)$. Define $\psi := \varphi^\vee$. Then ψ satisfies the admissibility condition (2.2.9). Hence, H is admissible. □

Question 8. Classify the pairs (H, M) satisfying the condition FO.

We will discuss the construction of frames for $L^2_M(\mathbb{R}^n)$ for pairs (H, M) satisfying the condition OF.

Lemma 3.2. *Let $H \subset \mathrm{GL}(n, \mathbb{R})$ be a closed subgroup such that H can be written as $H = ANR = NAR = RAN$ with R compact, A simply connected abelian, and such that the map*

$$N \times A \times R \ni (n, a, r) \mapsto nar \in H$$

is a diffeomorphism. Assume furthermore that R and A commute, and that R and A normalize N. Finally assume that there exists a co-compact discrete subgroup $\Gamma_N \subset N$. Let $\Gamma_A \subset A$ be a co-compact subgroup in A. Choose bounded measurable subsets $\mathbb{F}_A \subset A$, and $\mathbb{F}_N \subset N$ such that $N = \Gamma_N \mathbb{F}_N$, and $A = \Gamma_A \mathbb{F}$ and such that the union is disjoint. Let $\Gamma = \Gamma_A \Gamma_N$ and $\mathbb{F}_H = \mathbb{F}_N \mathbb{F}_A R \subset H$. Then we have

$$H = \bigcup_{\gamma \in \Gamma} \gamma \mathbb{F}_H$$

and the union is disjoint. Furthermore we can choose $\mathbb{F}_H$ such that $\mathbb{F}_H^o \not\emptyset$.

Proof. We have

$$\begin{aligned} \bigcup_{\gamma} \gamma \mathbb{F}_N \mathbb{F}_A R &= \Gamma_A \Gamma_N (\mathbb{F}_N) \mathbb{F}_A R \\ &= \Gamma_A N \mathbb{F}_A R \qquad \text{because}\, \Gamma_N \mathbb{F}_N = N \\ &= \bigcup_{\gamma \in \Gamma_A} (\gamma N \gamma^{-1}) \gamma \mathbb{F}_A R \\ &= \bigcup_{\gamma \in \Gamma_A} N \gamma \mathbb{F}_A R \qquad \text{because}\, A \,\text{normalizes}\, N \\ &= N \Gamma_A \mathbb{F}_A R \\ &= NAR \\ &= H\,. \end{aligned}$$

Assume now that

$$\gamma_A \gamma_N f_N f_A r = \sigma_A \sigma_N g_N g_A s$$

for some $\gamma_A, \sigma_A \in \Gamma_A$, $\gamma_N, \sigma_N \in \Gamma_N$, $f_A, g_A \in \mathbb{F}_A$, $f_N, g_N \in \mathbb{F}_N$, and $r, s \in R$. Then, as $A \times N \times R \ni (a, n, r) \mapsto anr \in H$ is a diffeomorphism, it follows that $r = s$. Hence $\gamma_A \gamma_N f_N f_A = \sigma_A \sigma_N g_N g_A$. But then – as A normalizes N –

$$\begin{aligned} \gamma_N f_N &= (\gamma_A^{-1} \sigma_A) \sigma_N g_N (g_A f_A^{-1}) \\ &= (\gamma_A^{-1} \sigma_A g_A f_A^{-1}) \left((g_A f_A)^{-1} \sigma_N g_N (g_A f_A^{-1})\right) \end{aligned}$$

Hence $\gamma_A^{-1} \sigma_A g_A f_A^{-1} = 1$ or

$$\gamma_A f_A = \sigma_A g_A\,.$$

As the union $\Gamma_A \mathbb{F}_A$ is disjoint, it follows that $\gamma_A = \sigma_A$ and $f_A = g_A$. But then the above implies that

$$\gamma_N f_N = \sigma_N g_N\,.$$

But then - again because the union is disjoint – it follows that $\gamma_N = \sigma_N$ and $f_N = g_N$. □

Our first application of this lemma is to give a simple proof of the main result, Theorem 4.2, of [51], without using the results of [8]. We will reformulate those results so as to include sampling on irregular grids, see also [4].

Assume now that $H = ANR$ satisfies the conditions in Lemma 3.2. Let $\Gamma = \Gamma_A \Gamma_N$ and $\mathbb{F}_H = \mathbb{F}_N \mathbb{F}_A R$ be as in that Lemma. Suppose that $M \subseteq \mathbb{R}^n$ is H invariant and such that there are finitely many open orbits $U_1, \ldots, U_k \subseteq M$ such that $|M \setminus \bigcup_{j=1}^k U_j| = 0$. Finally we assume that for each $\omega_j \in U_j$ the stabilizer of ω_j in H under the action $(h, \omega) \mapsto (h^{-1})^T(\omega)$ is contained in R and hence compact. Let $\mathbb{F}_j = F_H \cdot \omega_j$ and $\mathbb{F} = \bigcup_{j=1}^k \mathbb{F}_j$. Then the Lemma 3.2 implies that we have a multiplicative tiling of M as

$$M = \bigcup_{\gamma \in \Gamma} \gamma \cdot \mathbb{F} = \bigcup_{\gamma \in \Gamma^{-1}} \gamma^T(\mathbb{F}) \tag{3.3.1}$$

(up to set of measure zero).

Theorem 3.3. *Let the notation be as above. Suppose that $\{e_t|_{\mathbb{F}}\}_{t \in \mathcal{T}}$ is a frame for $L^2(\mathbb{F})$. Let $\psi = \chi_{\mathbb{F}}^{\vee}$. Then the sequence $\{\pi((t,\gamma)^{-1})\psi\}_{(t,\gamma) \in \mathcal{T} \times \Gamma}$ is a frame for $L^2(M)$ with the same frame bounds. In particular there exists a constant $T > 0$ such that $\{T\pi((t,\gamma)^{-1})\psi\}_{(t,\gamma) \in \mathcal{T} \times \Gamma}$ is a Parseval frame for $L^2_M(\mathbb{R}^n)$ if and only if $\{Te_t|_{\mathbb{F}}\}_{t \in \mathcal{T}}$ is a Parseval frame for $L^2(\mathbb{F})$.*

Proof. This follows from Theorem 2.14. □

There are several ways to state different versions of the above theorem. In particular one can have different groups $H_j = A_j N_j R_j$, with compact stabilizers, such that each of them has finitely many open orbits, $U_{j,1}, \ldots, U_{j,k_j}$ such that $\mathbb{R}^n = \bigcup_{j,l} \Gamma_j U_{j,l}$ a disjoint union. One can also construct a set $\mathbb{F}_H$ that is not necessarily R-invariant (c.f. Example 3.8), and finally, one can allow more than three group. But we will not state all those obvious generalizations, but only notice the following construction from [21]. We refer to the Appendix for more details. In [21] the authors started with a prehomogeneous vector space (L, V) of parabolic type, see [9] for details. Then L has finitely many open orbits in V, but in general the stabilizer of a point is not compact. To deal with that, the authors constructed for each orbits U_j a subgroup $H_j = A_j N_j R_j$ such that U_j is up to measure zero a disjoint union of open H_j orbits $U_{j,i}$. It turns out, that it is not necessary to pick a different group for each orbit, the same group $H = H_j$ works for all the orbits.

Theorem 3.4. *Let $H = ANR$ be one of the group constructed in* [21]*. Then H is admissible.*

Proof. This follows from Theorem 3.6.3 and Corollary 3.6.4 in [9]. □

Remark 3.5. The statement in [9] is in fact stronger than the above remark. In most cases the group AN has finitely many open orbits. This group acts freely and is therefore admissible. The only exception is the so-called Type III spaces, where the group ANR has one orbit and is admissible.

Example 3.6 ($\mathbb{R}^+\mathrm{SO}(n)$)**.** Take $A = \mathbb{R}^+\mathrm{id}$, $R = \mathrm{SO}(n)$, the group of orientation preserving rotations in $\mathbb{R}^n$, and $N = \{\mathrm{id}\}$. Then $H = \mathbb{R}^+\mathrm{SO}(n)$ is the group of dilations and orientation preserving rotations. Notice that $g^{-1} = g^T$ if $g \in \mathrm{SO}(n)$ and therefore $g \cdot \omega = g(\omega)$. The group H has two orbits $\{0\}$ and $\mathbb{R}^n \setminus \{0\}$. The stabilizer of $e_1 = (1, 0, \ldots, 0)^T$ is isomorphic to $\mathrm{SO}(n-1)$. In particular the stabilizer group is compact. It follows that $\mathbb{R}^+\mathrm{SO}(n)$ is admissible. In fact any function with compact support in $\mathbb{R}^n \setminus \{0\}$ is, up to normalization, the Fourier transform of an admissible wavelet.

Example 3.7 (Diagonal matrices)**.** Let H be the group of diagonal matrices $H = \{d(\lambda_1, \ldots, \lambda_n) \mid \lambda_j \neq 0\}$. Thus $A = \{d(\lambda_1, \ldots, \lambda_n) \mid \forall j : \lambda_j > 0\}$ and $R = \{d(\epsilon_1, \ldots, \epsilon_n) \mid \epsilon_j = \pm 1\}$. The group N is trivial. Then H has one open and dense orbit

$$U = \{(x_1, \ldots, x_n)^T \mid (j = 1, \ldots, n)\, x_j \neq 0\} = H \cdot (1, \ldots, 1)^T .$$

The stabilizer of $(1, \ldots, 1)^T$ is trivial and hence compact. It follows that H is admissible. We can also replace H by the connected group A. Then we have 2^n open orbits parameterized by $\epsilon \in \{-1, 1\}^2$

$$U_\epsilon = \{(x_1, \ldots, x_j)^T \mid \mathrm{sign}(x_j) = \epsilon_j\} = H \cdot (\epsilon_1, \ldots, \epsilon_n) .$$

The stabilizers are still compact and hence H is admissible.

Example 3.8 (Some two-dimensional examples)**.** In this example we discuss some 2-dimensional applications. Most of those examples can be found in several other places in the literature, but we would like to show how all of them fits into our general construction.

Upper triangular matrices: Let first H be the group of upper triangular 2×2-matrices of determinant 1,

$$H = \left\{ \begin{pmatrix} a & t \\ 0 & 1/a \end{pmatrix} \mid a \neq 0,\, t \in \mathbb{R} \right\}.$$

Here A is the group of diagonal matrices with $a > 0$, N is the group up upper triangular matrices with 1 on the main diagonal and $R = \{\pm \mathrm{id}\}$. Then we have one open orbit of full measure given by

$$U = \{(x,y)^T \mid y \neq 0\} = H \cdot e_2$$

where $e_2 = (0,1)^T$. The stabilizer of e_2 is trivial which implies that H is admissible.

Now take $R = \{\mathrm{id}\}$. Then AN has two open orbits

$$U_1 = \{(x,y)^T \mid y > 0\} \quad \text{and} \quad U_2 = \{(x,y)^T \mid y < 0\}.$$

Take any $\lambda > 1$ and $b > 0$. Then we can take

$$\Gamma_N := \left\{ \begin{pmatrix} 1 & kb \\ 0 & 1 \end{pmatrix} \mid k \in \mathbb{Z} \right\} \quad \text{and} \quad \Gamma_A := \left\{ \begin{pmatrix} \lambda^n & 0 \\ 0 & \lambda^{-n} \end{pmatrix} \mid n \in \mathbb{Z} \right\}.$$

Take

$$\mathbb{F}_N = \left\{ \begin{pmatrix} 1 & kb \\ 0 & 1 \end{pmatrix} \mid k \in \mathbb{Z} \right\} \quad \text{and} \quad \Gamma_A := \left\{ \begin{pmatrix} \mu & 0 \\ 0 & 1/\mu \end{pmatrix} \mid \mu \in (1/\lambda, 1) \right\}.$$

Taking $\omega = (0,1)^T$ we get

$$\mathbb{F} = \{(x,y)^T \in \mathbb{R}^2 \mid x = ty \text{ for some } t \in (0,b) \text{ and some } y \in (1/\lambda, 1)\}.$$

There are several choices for a spectral set Ω containing $\mathbb{F}$. One of them is

$$\Omega = \{(x,y)^T \mid 0 < x < b,\, \frac{1}{\lambda} < y < 1\}.$$

To see how the elements in Γ acts, lets us see how the generators of Γ_N and Γ_A acts:

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mathbb{F}\{(x,y)^T \in \mathbb{R}^2 \mid x = ty,\, t \in (b, 2b),\, y \in (1/\lambda, 1)\}$$

In particular, the line segment that bound $\mathbb{F}$ are moved into:

$$\begin{aligned}
\{(x,1)^T \mid 0 < x < b\} &\quad\mapsto\quad \{(x,1) \mid b < x < 2b\} \\
\{(x,1/\lambda)^T \mid 0 < x < b/\lambda\} &\quad\mapsto\quad \{(x,1/\lambda) \mid b/\lambda < x < 2b/\lambda\} \\
\{(0,y)^T \mid \frac{1}{\lambda} < y < 1\} &\quad\mapsto\quad \{(x,y) \mid x = by,\, \frac{1}{\lambda} < y < 1\} \\
\{(x,y)^T \mid x = by,\, \frac{1}{\lambda} < y < 1\} &\quad\mapsto\quad \{(x,y) \mid x = 2by,\, \frac{1}{\lambda} < y < 1\}
\end{aligned}$$

Similarly

$$\begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix} \mathbb{F} = \{(x,y)^T \in \mathbb{R}^2 \mid x = sy \text{ for some } s \in (0, \lambda^2 b) \text{ and some } y \in (\lambda^{-2}, \lambda^{-1})\}.$$

Dilations and rotations: In this case we let $A = \mathbb{R}^+\mathrm{id}$, the group of dilations,

$$N = \mathrm{SO}_o(2) = \left\{ R_\theta \begin{pmatrix} \cos(2\pi t) & \sin(2\pi t) \\ -\sin(2\pi t) & \cos(2\pi t) \end{pmatrix} \mid 0 \leq t \leq 1 \right\},$$

the group of orientation preserving rotations, and $R = \{\mathrm{id}\}$. (Notice, that we could have interchanged to role of R and N, as N is compact in this case.) Then we have one open orbit $U = \mathbb{R}^2 \setminus \{(0,0)^T\}$. Fix $\lambda > 1$ and $s \in \mathbb{N}$ and take $\Gamma_A = \{\lambda^n \mathrm{id} \mid n \in Z\}$ and $\Gamma_N = \{R_{k/s} \mid k = 0, 1, \ldots, s-1\}$. Then Γ_N is a finite subgroup of N and $\Gamma_A \Gamma_N$ is a group. In this case we can take

$$\mathbb{F} = \{r(\cos(2\pi\theta), \sin(2\pi\theta)^T \mid \lambda^{-1} < r < 1,\, \theta \in (-1/(2s), 1/(2s))\} .$$

Dilations and hyperbolic rotations: The example $H = \mathbb{R}^+\mathrm{SO}_o(1,n)$ was discussed in details in [51]. Take $n = 2$. Then we can take $A = \mathbb{R}^+\mathrm{id}$ and and

$$N = \left\{ \begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix} \mid t \in \mathbb{R} \right\} .$$

In this case we have four open orbits

$$\begin{aligned} U_1 &= \{(x,y)^T \mid 0 < |y| < x\}, \quad U_2 = \{(x,y)^T \mid 0 < |y| < -x\}, \\ U_3 &= \{(x,y)^T \mid 0 < |x| < y\}, \quad U_4 = \{(x,y)^T \mid 0 < |x| < -y\} . \end{aligned}$$

Fix $\lambda > 1$ and $b > 0$. Let $\Gamma_A := \{\lambda^n \mathrm{id} \mid n \in \mathbb{Z}\}$ and

$$\Gamma_N := \left\{ \begin{pmatrix} \cosh(kb) & \sinh(kb) \\ \sinh(kb) & \cosh(kb) \end{pmatrix} \mid k \in \mathbb{Z} \right\} .$$

Notice, that in this case Γ is in fact a group as A and N commute. In this case we can take

$$\mathbb{F}_A = \{\mu \mathrm{id} \mid 1 < \mu < \lambda\} \quad \text{and} \quad \mathbb{F}_N = \left\{ \begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix} \mid t \in (-b/2, b/2) \right\} .$$

The fundamental domain in U_1 is then

$$\mathbb{F} = \{(x,y)^T \mid 1 < x < \cosh(b/2),\ x^2 - y^2 \in (1, \lambda^2)\} .$$

The boundary of this domain is given by

$$\begin{aligned} \partial\mathbb{F} &= \{(\cosh(t), \sinh(t))^T \mid -b/2 < t < b/2\} \cup \{\lambda(\cosh(t), \sinh(t))^T \mid -b/2 < t < b/2\} \\ &\quad \cup \{s(\cosh(b/2), \pm\sinh(b/2) \mid 1 \leq s \leq \lambda\} . \end{aligned}$$

We leave it to the reader to determine a spectral pair $(\Omega, \mathcal{T})$ such that $\mathbb{F} \subset \Omega$.

4. Construction of wavelet sets

We apply now our construction in section 1 to discrete subgroups of $\mathrm{GL}(n,\mathbb{R})$. We start by the following reformulation of Theorem 2.8 for discrete groups. Our aim is later to apply it to the discrete subgroup Γ_A from the last section. As before we use the notation $a \cdot x(a^{-1})^T(x)$.

Lemma 4.1. *Let D be a discrete subgroup of $\mathrm{GL}(n,\mathbb{R})$. If for almost every $x \in \mathbb{R}^n$, there exists an $\epsilon > 0$ such that ϵ-stabilizer D_ϵ^x is finite, then there exists a measurable function h such that*

$$\sum_{d \in D} |h(d^T x)|^2 = 1 \quad a.e. \tag{4.4.1}$$

We have the following improvement of Lemma 4.1.

Proposition 4.2. *Let D be a discrete subgroup of $\mathrm{GL}(n,\mathbb{R})$. If for almost every $x \in \mathbb{R}^n$, there exists an $\epsilon > 0$ such that D_ϵ^x is finite, then there exists a measurable function g of the form $g = \chi_K$ such that*

$$\sum_{d \in D} |g(d^T x)|^2 = 1 \quad a.e. \tag{4.4.2}$$

Proof. We first recall some notation and preliminary results from [46]. For an open ball $B \subset \mathbb{R}^n$, we define the orbit density function $f_B : \mathbb{R}^n \to [0,\infty]$ by

$$f_B(x) = \mu(\{d \in D \mid d^T x \in \overline{B}\}), \tag{4.4.3}$$

where μ is counting measure. Lemma 2.6 of [46] asserts that

$$\Omega_0 := \{x \in \mathbb{R}^n \mid D_\epsilon^x \text{ non} - \text{compact } \forall \epsilon > 0\} = \{x \in \mathbb{R}^n \mid f_B(x) = \infty,\ \forall B : B \cap \mathcal{O}_x \neq \emptyset\} \tag{4.4.4}$$

Now, let $\mathcal{B} = \{B_j\}$, $j \in \mathbb{N}$, be an enumeration of the balls in $\mathbb{R}^n$ having rational centers and positive rational radii. Let $f_j = f_{B_j}$. We claim that

$$\mathbb{R}^n = \bigcup_{j \geq 1} \{x \in \mathbb{R}^n \mid f_j(x) = 1\} \bigcup \Omega_0 \bigcup N, \tag{4.4.5}$$

where

$$N := \bigcup_{d \in D, d \neq \mathrm{id}} \{x \in \mathbb{R}^n \mid d^T x = x\}. \tag{4.4.6}$$

To see this, suppose that $x \notin (\Omega_0 \bigcup N)$. Then, there exists an open ball B such that $B \cap \mathcal{O}_x \not\emptyset$, and $f_B(x) < \infty$. Since $B \cap \mathcal{O}_x \neq \emptyset$, there is a $d_0 \in D$ such that $d_0^T x \in B$. Therefore, there is a j such that $d_0^T x \in B_j \subset B$; in particular $B_j \cap \mathcal{O}_x \neq \emptyset$ and $\infty > f_j(x) > 0$. Now, write $\{d \in D \mid\ d^T x \in \overline{B_j}\}\{d_0^T, \ldots d_k^T\}$. Since $x \notin N$, the $d_i^T x = d_j^T x$ only if $i = j$. Hence, there is an open set O such that $\mu(\{d \in D \mid d^T x \in O\}1$. Choose j such that $d_0^T x \in B_j \subset O$ so that $f_j(x) = 1$.

Continuing along the lines of [46], let

$$\Omega_1 = \{x \in \mathbb{R}^n \mid f_1(x) = 1\}$$

and

$$\Omega_j = \{x \in \mathbb{R}^n \mid f_j(x) = 1\} \setminus (\Omega_1 \bigcup \cdots \bigcup \Omega_{j-1})$$

The sets $\{\Omega_j\}_{j \geq 1}$ form a disjoint collection of Borel sets such that $\mathbb{R}^n \setminus (\bigcup_{j=1}^\infty \Omega_j)$ has measure 0 (it is a subset of $\Omega \bigcup N$). Let us define

$$g(x) = \sum_{j=1}^\infty \chi_{\Omega_j}(x) \chi_{\overline{B_j}}(x) \tag{4.4.7}$$

and $g(x) = 0$ for $x \notin (\bigcup_{j \geq 1} \Omega_j)$. Note that

$$g(x) = \chi_K\,, \quad K = \bigcup_{j=1}^\infty (\Omega_j \cap \overline{B_j}), \tag{4.4.8}$$

so all that is needed to complete the proof, is to show $\{d^T K \mid d \in D\}$ is a tiling of $\mathbb{R}^n$, equivalently, $\sum_{d \in D} g(d^T x) = 1$ a.e. This is a special case of the argument in [46], which we outline now.

First, note that if $x \in \mathbb{R}^n$ such that $f_j(x) = 1$ for some smallest j, then there is a unique $d \in D$ such that $dx \in \overline{B_j}$. Since f_j is constant on orbits, $d^T x \in \Omega_j \cap \overline{B_j}$ and $d^T x \notin (\Omega_1 \bigcup \cdots \bigcup \Omega_{j-1})$. Therefore, $x \in d \cdot K$ and $\bigcup_{d \in D} d^T K = \mathbb{R}^n$ up to a set of measure 0.

For disjointness, since $d^T \Omega_j = \Omega_j$, it suffices to check that $(\Omega_j \cap \overline{B_j}) \cap (\Omega_j \cap \overline{B_j})a$ has measure 0 for all d not the identity id. If $x \in (\Omega_j \cap \overline{B_j})$, then $f_j(x)1$. If, in addition, $x = d^T \omega$ for some $\omega \in (\Omega_j \cap \overline{B_j})$, then $d^{-1} x \in \overline{B_j}$ which means that $d \cdot x = x$ and $d = \mathrm{id}$ since $f_j(x) = 1$. $\square$

Theorem 4.3. *Let D be a discrete subgroup of $\mathrm{GL}(n, \mathbb{R})$ that contains an expansive matrix, and $\mathcal{L} \subset \mathbb{R}^n$ a full rank lattice. If for almost every $x \in \mathbb{R}^n$, there exists an $\epsilon > 0$ such that D_ϵ^x is finite, then, there exists a $(D, \mathcal{L})$-wavelet set.*

Proof. By Proposition 4.2, there exists a function $g = \chi_K$ such that equation 4.4.2 holds. Therefore, $\{d^T K \mid d \in D\}$ tiles $\mathbb{R}^n$. Thus, by Theorem 1.17 there exists a $(D, \mathcal{L})$-wavelet set. □

Note that there are examples of discrete subgroups of $\mathrm{GL}(n, \mathbb{R})$ that are not generated by a single element, c.f. Example 3.8.

Question 9. If $\mathcal{D}$ is a subgroup of $\mathrm{GL}(n, \mathbb{R})$ and $\mathcal{L}$ is a full rank lattice such that there exists a $(\mathcal{D}, \mathcal{T})$-wavelet set, then for almost every x does there exist an $\epsilon > 0$ such that the ϵ-stabilizer D^x_ϵ is finite?

One final comment is that in all of the above considerations, the set D is assumed to be invariant under multiplication by an expansive matrix. Removing this condition seems to be very hard. Indeed, even when the set $D = \{a^j \mid j \in \mathbb{Z}\}$, it is not clear what happens when a is not an expansive matrix. In this case, the interplay between dilations and translations becomes crucial in understanding when there exists a wavelet set. For example, let $a = \begin{pmatrix} 2 & 0 \\ 0 & 2/3 \end{pmatrix}$, $D = \{a^j \mid j \in \mathbb{Z}\}$, and $\mathcal{T}\mathbb{Z}^2$. It is easy to see that there is a set of finite measure Ω such that $\{a^j \Omega\}$ tiles $\mathbb{R}^2$. However, there exist full rank lattices $\mathcal{L}_1$ and $\mathcal{L}_2$ such that there are no $(D, \mathcal{L}_1)$-wavelet sets, yet there are $(D, \mathcal{L}_2)$-wavelet sets [54]. Hence, in the non-expansive case, it is not enough to simply prove the existence of sets that tile via dilations and translations separately.

We will now apply this to the discrete subgroup $\Gamma_A \subset A$ from the last section, where A is as in Theorem 3.4.

Theorem 4.4. *Let $H = ANR$ be one of the group constructed in* [21]. *Then the following holds:*

1. *The group A contains a discrete co-compact subgroup $\Gamma_A \subset A$ such that $E_A = \{d \in \Gamma_A \mid d \text{ is expansive}\}$ is a non-trivial subsemigroup. In particular there exists an expansive matrix a such that Γ_A is a invariant.*
2. *Let Γ_A and Γ_A^+ be as in the Appendix and let $\Gamma = \Gamma_A \Gamma_N$. Then $\Gamma \Gamma_A^+ \subset \Gamma$.*

Proof. (1) follows from Lemma 5.3 in the appendix. For (2) we recall that Γ_A^+ is contained in the center of ANR. Hence

$$\begin{aligned} \Gamma \Gamma_A^+ &= \Gamma_A \Gamma_N \Gamma_A^+ \\ &= \Gamma_A \Gamma_A^+ \Gamma_N \\ &\subseteq \Gamma_A \Gamma_N = \Gamma \,. \end{aligned}$$

□

We have now proved, using Theorem 4.3 the following theorem:

Theorem 4.5. *Let the notation be as in Theorem 4.4. Let $\mathcal{L}$ be a full rank lattice in $\mathbb{R}^n$. Then there exists a $(\Gamma_A, \mathcal{L})$-wavelet set and a $(\Gamma, \mathcal{L})$-wavelet set.*

Remark 4.6. Theorem 4.4 gives several examples of non-groups of dilations for which wavelet sets exist. Unfortunately from the point of view of characterizing sets $(\mathcal{D}, \mathcal{T})$ for which wavelet sets exist, if one starts with the set $\mathcal{D}$, one still has to rely on the existence of an object external to the set $\mathcal{D}$ for the existence of wavelet sets. It would also be interesting to remove the condition that $\mathcal{L}$ is a lattice.

5. Symmetric cones

In this section we discuss the important example of homogeneous cones in $\mathbb{R}^n$. Those cones show up in several places in analysis. As an example one can take Hardy spaces of holomorphic function on tube type domains $\mathbb{R}^n + i \oplus \Omega$ [55]. An excellent reference for harmonic analysis on symmetric cones is the book by J. Faraut and A. Koranyi [22]. A nonempty open subset $\Omega \subset \mathbb{R}^n$ is called an *open (convex) cone* if Ω is convex and $\mathbb{R}^+\Omega \subseteq \Omega$. Let Ω be an open cone, define the *dual cone* Ω^* by

$$\Omega^* := \{v \in \mathbb{R}^n \mid \forall u \in \overline{\Omega} \setminus \{0\} \ : \ (v,u) > 0\} \ .$$

If Ω^* is nonempty, then Ω^* is a open cone. Ω is *self-dual* if $\Omega = \Omega^*$. Let

$$\mathrm{GL}(\Omega) = \{g \in \mathrm{GL}(n,\mathbb{R}) \mid g(\Omega) = \Omega\} \ .$$

Then Ω is *homogeneous* if $\mathrm{GL}(\Omega)$ acts transitively on Ω. From now on we assume that Ω is a self-dual homogeneous cone. Let $g \in \mathrm{GL}(\Omega)$ and $u \in \overline{\Omega} \setminus \{0\}$. Then $g(u) \in \overline{\Omega} \setminus \{0\}$. Hence if $v \in \Omega = \Omega^*$, then

$$(g^T(v), u) = (v, g(u)) > 0 \ .$$

Thus $g^T(v) \in \Omega^* = \Omega$. It follows that $\mathrm{GL}(\Omega)$ is invariant under transposition, and hence reductive. Let $e \in \Omega$. Then

$$K = \mathrm{GL}(\Omega)^e = \{g \in \mathrm{GL}(\Omega) \mid g(e) = e\} \ .$$

Let $\theta(g) = (g^{-1})^T$. Then it is always possible to choice e such that $K = \{g \in \mathrm{GL}(\Omega) \mid \theta(g) = g\} = \mathrm{SO}(n) \cap \mathrm{GL}(n,\mathbb{R})$. Define the *Lie algebra* of $\mathrm{GL}(\Omega)$ by

$$\mathfrak{gl}(\Omega) := \{X \in M(n,\mathbb{R}) \mid \forall t \in \mathbb{R} \ : \ e^{tX} \in \mathrm{GL}(\Omega)\} \ .$$

Then $\mathfrak{gl}(\Omega)$ is invariant under the Lie algebra automorphism $\dot\theta(X) = -X^T$. Let

$$\mathfrak{k} = \{X \in \mathfrak{gl}(\Omega) \mid \dot\theta(X) = X\}$$

and

$$\mathfrak{s} = \{X \in \mathfrak{gl}(\Omega) \mid \dot\theta(X) = -X\} = \mathrm{Symm}(n,\mathbb{R}) \cap \mathfrak{gl}(\Omega)$$

where $\mathrm{Symm}(n,\mathbb{R})$ stand for the space of symmetric matrices. Let $\mathfrak{a}$ be a maximal subspace in $\mathfrak{s}$ such that $[X,Y] = XY - YX = 0$ for all $X, Y \in \mathfrak{a}$. Notice that $(X,Y) = \mathrm{Tr}(XY^T)$ is an inner product on $\mathfrak{gl}(\Omega)$ and that, with respect to this inner product, $\mathrm{ad}(X) : \mathfrak{gl}(\Omega) \to \mathfrak{gl}(\Omega)$, $Y \mapsto [X,Y]$ satisfies

$$\mathrm{ad}(X)^T = \mathrm{ad}(X^T) \ .$$

Hence the algebra $\{\mathrm{ad}(X) \mid X \in \mathfrak{a}\}$ is a commuting family of self adjoint operator on the finite dimensional vector space $\mathfrak{gl}(\Omega)$. Hence there exists a basis $\{X_j\}_j$ of $\mathfrak{gl}(\Omega)$ consisting of joint eigenvectors of $\{\mathrm{ad}(X) \mid X \in \mathfrak{a}\}$. Let $\mathfrak{z}(\mathfrak{a})$ be the zero eigenspace, i.e., the maximal subspace of $\mathfrak{gl}(\Omega)$ commuting with all $X \in \mathfrak{a}$. Then there exists a finite subset $\Delta \subset \mathfrak{a}^* \setminus \{0\}$ such that with

$$\mathfrak{gl}(\Omega)^\alpha = \{Y \in \mathfrak{gl}(\Omega) \mid \forall X \in \mathfrak{a} \ : \ \mathrm{ad}(X)Y = \alpha(X)Y\}$$

we have

$$\mathfrak{gl}(\Omega) = \mathfrak{z}(\mathfrak{a}) \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{gl}(\Omega)^\alpha \ .$$

Notice that if $\alpha \in \Delta$ then $-\alpha \in \Delta$. In fact, if

$$X \in \mathfrak{gl}(\Omega)^\alpha \Longrightarrow X^T \in \mathfrak{gl}(\Omega)^{-\alpha} \ . \tag{5.5.1}$$

Let $\mathfrak{a}' = \{X \in \mathfrak{a} \mid \forall \alpha \in \Delta \ : \ \alpha(X) \neq 0\}$. Then $\mathfrak{a}'$ is open and dense in $\mathfrak{a}$. In particular $\mathfrak{a}' \neq \{0\}$. Fix $Z \in \mathfrak{a}'$ and let $\Delta^+ = \{\alpha \in \Delta \mid \alpha(Z) > 0\}$. Then $\Delta = \Delta^+ \bigcup -\Delta^+$, and if $\alpha, \beta \in \Delta^+$ are such that $\alpha + \beta \in \Delta$, then $\alpha + \beta \in \Delta^+$. Let

$$\mathfrak{n} = \bigoplus_{\alpha \in \Delta^+} \mathfrak{gl}(\Omega)^\alpha.$$

Then $\mathfrak{n}$ is a nilpotent Lie algebra (as $[\mathfrak{gl}(\Omega)^\alpha, \mathfrak{gl}(\Omega)^\beta] \subset \mathfrak{gl}(\Omega)^{\alpha+\beta}$) and $[\mathfrak{a}, \mathfrak{n}] \subseteq \mathfrak{n}$. In particular it follows that $\mathfrak{q} = \mathfrak{a} \oplus \mathfrak{n}$ is a solvable Lie algebra. Notice that the algebra $\mathfrak{z}(\mathfrak{a})$ is invariant under transposition. Hence $\mathfrak{z}(\mathfrak{a}) = \mathfrak{z}(\mathfrak{a}) \cap \mathfrak{k} \oplus \mathfrak{a}$. Because of (5.5.1) it therefore follows that

$$\mathfrak{gl}(\Omega) = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n} \ .$$

This decomposition is called *the Iwasawa decomposition of* $\mathfrak{gl}(\Omega)$. Let $A = \{e^X \mid X \in \mathfrak{a}\}$ and $N = \{e^Y \mid Y \in \mathfrak{n}\}$. Then A and N are Lie groups, A is abelian, and $aNa^{-1} = N$ for all $a \in A$. It follows that $Q := AN = NA$ is a Lie group with N a normal subgroup. Furthermore we have the following *Iwasawa decomposition of* $\mathrm{GL}(\Omega)$:

Lemma 5.1 (The Iwasawa decomposition)**.** *The map*

$$A \times N \times K \ni (a, n, k) \mapsto ank \in \mathrm{GL}(\Omega)$$

is an analytic diffeomorphism.

We note that the one dimensional group $Z = \mathbb{R}^+\mathrm{id}$ is a subgroup of $\mathrm{GL}(\Omega)$ and in fact $Z \subset A$. If $a(\lambda) = \lambda\mathrm{id} \in Z$, with $\lambda > 1$, then $a(\lambda)$ is expansive. In particular it follows that the set E of expansive matrices in A is a nonempty subsemigroup of A. Let $X_0 = \mathrm{id}$, $X_1, \dots, X_r$ be a basis of $\mathfrak{a}$ and let

$$\Gamma_A = \{\exp(n_0 X_0 + \ldots + n_r X_r) \mid n_j \in \mathbb{Z}\} \ .$$

Then A/Γ_A is compact. Furthermore there exists a discrete subgroup $\Gamma_N \subset N$ such that N/Γ_N is compact.

Let now $\mathcal{D} = \Gamma$ and $d = \exp(2X_0)$. Then d is expansive and $d\mathcal{D} = \mathcal{D}d \subset \mathcal{D}$, because d is central in $\mathrm{GL}(\Omega)$. It follows that the results from the previous sections are applicable in this case.

Appendix: Prehomogeneous vector spaces

One way to find admissible groups with finitely many open orbits is to start with *prehomogeneous vector spaces*. Those are pairs (H, V) where H is a reductive Lie group, say $H^T = H$, and V is a finite dimensional vector space, such that H has finitely many open orbits in V. There is no full classification of those spaces at the moment, but a subclass, the *prehomogeneous vector spaces of parabolic type* has been classified. We refer to [9] Section 2.11, for detailed discussion and references. The problem, from the point of view of our work is, that the compact stabilizer condition does not hold in general, but as shown in [21] one can always replace H by a subgroup of the form ANR as before, such that ANR is admissible. Notice that, by using either ANR or $A^T N^T R^T$, which satisfies the same conditions, we can consider either the standard action on $\mathbb{R}^n$ or the action $(a, x) \mapsto (a^{-1})^T(x)$. We will use the second action in what follows.

Let $H = H^T$ be a reductive Lie group acting on $V = \mathbb{R}^n$. Then H can be written as $H = LC$ where $C = C^T$ is a vector group, isomorphic to a abelian subalgebra $\mathfrak{c}$ of $\mathfrak{gl}(n, \mathbb{R}) = M(n, \mathbb{R})$. The isomorphism is simply given by the matrix exponential function

$$X \mapsto \exp(X) = e^X = \sum_{j=0}^{\infty} \frac{X^j}{j!} \ .$$

The vector space V is graded in the sense that there exists a subset $\Delta \subset \mathfrak{c}^*$ such that

$$V = \bigoplus_{\alpha \in \Delta} V_\alpha \tag{5.5.2}$$

where

$$V_\alpha = \{v \in V \mid (\forall \in \mathfrak{c}) \, : \, X \cdot v = \alpha(X)v\} \ . \tag{5.5.3}$$

If $c = \exp(X) \in C$ and $\lambda \in \mathfrak{c}^*$, then we write $c^\lambda = e^{\lambda(X)}$. In particular $c \cdot v = c^\alpha v$ for all $v \in V_\alpha$.

Denote by pr_α the projection onto V_α along $\bigoplus_{\beta \neq \alpha} V_\beta$.

Lemma 5.2. *We have $H \cdot V_\alpha \subset V_\alpha$ for all $\alpha \in \Delta$. Furthermore if $v \in V$ and $H \cdot v$ is open, then $\mathrm{pr}_\alpha(v) \neq 0$ for all $\alpha \in \Delta$.*

Proof. Let $c \in C$, $h \in H$ and $v \in V_\alpha$. As C is central in C it follows that $c \cdot (h \cdot v) = h \cdot (c \cdot v) = c^\alpha h \cdot v$. □

The set Δ has the properties that $0 \notin \Delta$, if $\alpha \in \Delta$, then $-\alpha \notin \Delta$, and finally there exists $\alpha_1, \ldots, \alpha_k \in \Delta$ such that if $\alpha \in \Delta$, then there are $n_1, \ldots, n_r \in \mathbb{N}_0$ such that

$$\alpha = n_1\alpha_1 + \ldots + n_r\alpha_r \,. \tag{5.5.4}$$

For $\alpha \in \Delta$ let $\mathcal{N}_\alpha = \{X \in \mathfrak{c} \mid \alpha(X) = 0\}$. Then $\bigcup_{\alpha\in\Delta} \mathcal{N}_\alpha$ is a finite union of hyperplanes and hence the complement is open and dense in $\mathfrak{c}$. Let $\mathfrak{c}^+$ be a connected component of the complement of $\bigcup \mathcal{N}_\alpha$. Because of (5.5.4) we can choose $\mathfrak{c}^+$ such that

$$\forall X \in \mathfrak{c}^* \forall \alpha \in \Delta \,:\, \alpha(X) > 0 \,. \tag{5.5.5}$$

Notice that $\mathfrak{c}^+$ is convex, $\mathfrak{c}^+ + \mathfrak{c}^+ \subset \mathfrak{c}^+$ and $\mathbb{R}^+\mathfrak{c}^+ \subset \mathfrak{c}^+$.

Lemma 5.3. *The group A contains a non-trivial abelian semigroup of expanding matrices.*

Proof. Let $C^+ := \exp(\mathfrak{c}^+)$. Suppose that $a, b \in C^+$. Choose $X, Y \in \mathfrak{c}^+$ such that $a = \exp(X)$ and $b = \exp(Y)$. Then $ab = \exp(X + Y) \in C^+$. Thus C^+ is a semigroup. Let $a = \exp(X)$ be as above. Let $v = \sum_\alpha v_\alpha \in V$ with $v_\alpha \in$, then

$$\exp(X) \cdot v = \sum_\alpha e^{\alpha(X)} v_\alpha$$

and $e^{\alpha(X)} > 1$ because $\alpha(X) > 0$ for all α. □

Choice a basis $X_1, \ldots, X_r$ of $\mathfrak{a}$ such that the vectors $X_1, \ldots, X_k$ and the vectors $X_{k+1}, \ldots X_r$ form a basis for the orthogonal complement of $\mathfrak{c}$ in $\mathfrak{a}$. Here we use the inner product $(X, Y) = \mathrm{Tr}(XY^T)$. As $\mathfrak{c}^+$ is an open cone in $\mathfrak{c}$ we can choose $X_j \in \mathfrak{c}^+$, $j = 1, \ldots, k$. Let

$$\Gamma_A := \{\exp(\sum_{j=1}^r n_j X_j) \mid \forall j \,:\, n_j \in \mathbb{Z}\} \,.$$

Then Γ_A is a co-compact, discrete subgroup of A and every element of

$$\Gamma_A^+ := \{\exp(n_1 X_1 + \ldots + n_k X_k) \mid \forall j \,:\, n_j \geq 0\} \setminus \{\mathrm{id}\}$$

is expansive. As

$$\exp(n_1 X_1 + \ldots + n_r X_r) \exp(m_1 X_1 + \ldots + m_r X_r) \exp((n_1 + m_1) X_1 + \ldots + (n_r + m_r) X_r)$$

it follows that Γ_A^+ is a subsemigroup of Γ_A such that $\Gamma_A \Gamma_A^+ \subset \Gamma_A$. Thus Γ_A is γ invariant for all $\gamma \in \Gamma_A^+$. We have therefore shown the following:

Lemma 5.4. *There exists a co-compact discrete subgroup Γ of A and a subsemigroup Γ_A^+ such that each element of Γ_A^+ is expansive and $\Gamma_A \Gamma_A^+ \subset \Gamma_A$.*

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Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803, USA
E-mail address: `olafsson@math.lsu.edu`

Department of Mathematics, Saint Louis University, St. Louis, MO 63103
E-mail address: `speegled@slu.edu`