# A REDUNDANT VERSION OF THE RADO-HORN THEOREM

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Abstract. The Rado-Horn Theorem gives a characterization of those sets of vectors which can be written as the union of a fixed number of linearly independent sets. In this paper we study the redundant case. We show that then the span of the vectors can be written as the direct sum of a subspace which directly fails the Rado-Horn criteria and a subspace for which the Rado-Horn criteria hold. As a corollary, we characterize those sets of vectors which, after the deletion of a fixed number of vectors, can be written as the finite union of linearly independent sets.

## 1. Introduction

The Rado-Horn Theorem [1, 2] gives a characterization of vectors which can be written as the finite union of $M$ linearly independent sets.

**Theorem 1** (Rado-Horn). *Let $I$ be a countable index set, $\{f_i : i \in I\}$ be a collection of vectors in a vector space, and $M \in \mathbb{N}$. Then the following conditions are equivalent.*

(i) *There exists a partition $\{I_j : j = 1, \ldots, M\}$ such that for each $1 \leq j \leq M$, $\{f_i : i \in I_j\}$ is linearly independent.*

(ii) *For all finite $J \subset I$,*

$$\frac{|J|}{\dim \operatorname{span}\left(\{f_i : i \in J\}\right)} \leq M. \tag{1}$$

The terminology "Rado-Horn Theorem"was introduced, to our knowledge, in the paper [3]. This theorem has had at least two interesting applications in analysis; namely, a characterization of Sidon sets in $\Pi_{k=1}^{\infty}\mathbb{Z}_p$ [4, 5] and progress on the Feichtinger conjecture in [6]. There have also been at least three proofs, all in a similar spirit, of the Rado-Horn Theorem published [7, 2, 1]. Pisier, when discussing a characterization of Sidon sets in $\Pi_{k=1}^{\infty}\mathbb{Z}_p$ states "…d'un lemme d'algébre dû à Rado-Horn dont la démonstration est relativement délicate. [5, p. 704]"

In this paper, we prove a generalization of the Rado-Horn Theorem to the redundant case; that is, we consider the case that, after fixing $M \in \mathbb{N}$, the collection of vectors $\{f_i : i \in I\}$ cannot be partitioned into $M$ linearly independent sets. It is not hard to see (see Corollary 4) that the partition that maximizes the sum of the dimensions

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of the spans of the vectors is a partition into linearly independent sets in the case that the hypotheses of the Rado-Horn Theorem are satisfied. Our idea is to study the partition maximizing this sum in the case that the hypotheses of Rado-Horn are not satisfied. In particular, for this case, there must be some set $J \subset I$ such that (1) fails. We prove in Theorem 11 that then there is a partition of the vectors $\{f_i : i \in I\}$ that, in some sense, "tries" to be linearly independent. In particular, after partitioning, it becomes much easier to see which vectors are the obstacles to partitioning the set of vectors into linearly independent sets. Our proof is unfortunately no less delicate than the original proofs of the Rado-Horn Theorem, despite the addition of the idea of maximizing the sums of the dimensions of the partition at the beginning. However, we do obtain as a corollary to our main theorem the following formally stronger result than the Rado-Horn Theorem.

**Theorem 2.** *Let $I$ be a countable index set, $\{f_i : i \in I\}$ be a collection of vectors in a vector space, and $K, M \in \mathbb{N}$. Then the following conditions are equivalent.*

(i) *There exists a subset $H \subset I$ with $|H| = K$ such that $\{f_i : i \in I \setminus H\}$ can be written as the union of $M$ linearly independent sets.*
(ii) *For every finite $J \subset I$,*

$$\frac{|J| - K}{\dim \operatorname{span}(\{f_i : i \in J\})} \leq M. \tag{2}$$

This paper is organized as follows. In Section 2 we discuss our main idea of proof, state some preliminary results, and introduce the notion of a chain which will be employed heavily throughout. Section 3 contains two redundant versions of the Rado-Horn Theorem for a finite collection of vectors (Theorem 11 and Theorem 12). The proof of Theorem 2, which is the redundant version for an arbitrary countable collection of vectors, is then given in Section 4.

## 2. Preliminary results

We begin by fixing notation. All vectors will be assumed to be in an arbitrary vector space. Given a collection $\{f_i : i \in I\}$, and a subset $J \subset I$, we define $\mathbb{F}_J = \{f_i : i \in J\}$.

2.1. **Partitions that maximize the sum of dimensions.** The most difficult part of the proof of Theorem 1 is the finite case. So, our main concern is understanding and extending Theorem 1 in the finite case. To this end, our main idea is to partition $I$ into $\{I_j : j = 1, \dots, M\}$ that maximizes

$$\sum_{j=1}^{M} \dim \operatorname{span}(\mathbb{F}_{I_j}). \tag{3}$$

Using Theorem 1 it is an easy matter to show that, if it is possible to partition the set $\mathbb{F}_I$ into $M$ linearly independent sets, then the partition maximizing (3) does it.

**Proposition 3.** *Suppose $\{f_i : i \in I\}$ is a finite collection of vectors contained in a vector space, and $I$ is partitioned into sets $\{I_j : j = 1, \dots, M\}$. Then the following conditions are equivalent.*

(i) *For every* $j \in \{1, \dots, M\}$, $\mathbb{F}_{I_j}$ *is linearly independent.*
(ii) $\sum_{j=1}^{M} \dim \operatorname{span}(\mathbb{F}_{I_j}) = |I|$.

*Proof.* (i) $\Rightarrow$ (ii). Clearly, $\sum_{j=1}^{M} \dim \operatorname{span}(\mathbb{F}_{I_j}) = \sum_{j=1}^{M} |I_j| = |I|$.
(ii) $\Rightarrow$ (i). Note that

$$|I| = \sum_{j=1}^{M} \dim \operatorname{span}(\mathbb{F}_{I_j}) \leq \sum_{j=1}^{M} |I_j| = |I|.$$

Therefore, $\dim \operatorname{span}(\mathbb{F}_{I_j}) = |I_j|$ for each $1 \leq j \leq M$ and $\mathbb{F}_{I_j}$ is linearly independent. □

**Corollary 4.** *Given a finite collection of vectors* $\mathbb{F}_I$ *satisfying (1), if we partition* $I$ *into* $\{I_j : j = 1, \dots, M\}$ *such that (3) is maximized, then* $\mathbb{F}_{I_j}$ *is linearly independent for each* $1 \leq j \leq M$*.*

*Proof.* By applying Theorem 1, we obtain a partition $\{D_j : j = 1, \dots, M\}$ of $I$ such that each $\mathbb{F}_{D_j}$ is linearly independent. So,

$$|I| = \sum_{j=1}^{M} \dim \operatorname{span}(\mathbb{F}_{D_j}) \leq \sum_{j=1}^{M} \dim \operatorname{span}(\mathbb{F}_{I_j}) \leq |I|$$

Therefore, $\mathbb{F}_{I_j}$ is linearly independent for each $1 \leq j \leq M$ by Proposition 3. □

The following easy example gives some idea as to the difficulties involved in partitioning vectors into linearly independent sets.

**Example 5.** Let $f_1 = (1, 0), f_2 = (0, 1), f_3 = (1, 1)$, and $f_4 = (1, 1)$. Then, if one starts with the wrong linearly independent set, $\mathbb{F}_1 = \{f_1, f_2\}$, then one needs three sets to get each set linearly independent, while the alternative partition $\mathbb{F}_1 = \{f_1, f_3\}$, $\mathbb{F}_2 = \{f_2, f_4\}$ uses only two.

The next lemma will be needed in the proof of Theorem 11.

**Lemma 6.** *Let* $\{f_i : i \in I\}$ *be a finite collection of vectors in a vector space. Let* $M \in \mathbb{N}$ *and* $\{I_j : j = 1, \dots, M\}$ *be a partition of* $I$ *that maximizes* $\sum_{j=1}^{M} \dim \operatorname{span}(\mathbb{F}_{I_j})$ *over all partitions of* $I$*, and let* $p \in \{1, \dots, M\}$*. If* $f_k \in I_p$ *and* $f_k = \sum_{l \in I_p, l \neq k} \alpha_l f_l$*, then* $f_k \in \operatorname{span}(\mathbb{F}_{I_j})$ *for all* $1 \leq j \leq M$*.*

*Proof.* Assuming the hypothesis of the lemma, if $f_k = \sum_{l \in I_p, l \neq k} \alpha_l f_l$, then removing $f_k$ from $I_p$ keeps $\dim \operatorname{span}(\mathbb{F}_{I_p})$ constant. Since we know that $\{I_j : j = 1, \dots, M\}$ maximizes the sum of the dimensions of the spans, moving $f_k$ into another $I_j$, $j \neq p$ cannot increase $\dim \operatorname{span}(\mathbb{F}_{I_j})$, and the result follows. □

2.2. **Notion of a chain.** As part of the proof of our main theorem, we will be modifying our partition that maximizes (3) by moving linearly dependent vectors from one set to another. The following definition will be used to help us keep track of which vectors are being moved.

**Definition 7.** Let $\mathbb{F}_I = \{f_i : i \in I\}$ be a collection of vectors in a vector space. Let $\{I_j : j = 1, \dots, M\}$ be a partition of $I$ and let $L$ be a subset of $I_1$. We define a *chain of length $n$ starting in $L$ and ending at $a_n \in I$* to be a finite sequence $\{(a_1, b_1), \dots, (a_n, b_n)\}$, where $a_i \in I$ and $b_i \in \{1, \dots, M\}$, such that

- $a_1 \in L$,
- $b_1 = 1$,
- for $2 \le i \le n$, $a_i \in I_{b_i}$ and $f_{a_i} = \alpha f_{a_{i-1}} + \sum_{j \in I_{b_i}, j \neq a_i} \alpha_j f_j$ for some $\alpha \neq 0$, and
- $a_i \neq a_k$ for $i \neq k$.

A chain of length $n$ starting in $L$ and ending at $a_n \in I$ is a *chain of minimal length starting in $L$ and ending at $a_n$* if every chain starting in $L$ and ending at $a_n$ has length greater than or equal to $n$.

**Lemma 8.** *Let $(a_1, b_1), \dots, (a_n, b_n)$ be a chain of minimal length starting in $L$ and ending at $a_n$. Then, for each $1 \le i \le n$, $(a_1, b_1), \dots, (a_i, b_i)$ is a chain of minimal length starting in $L$ and ending at $a_i$.*

*Proof.* By induction it suffices to show that $(a_1, b_1), \dots, (a_{n-1}, b_{n-1})$ is a chain of minimal length. Suppose, for the sake of contradiction, that there did exist a chain $(u_1, v_1), \dots, (u_k, v_k)$ such that $u_k = a_{n-1}$ and $k < n - 1$. Since $(a_1, b_1), \dots, (a_n, b_n)$ is a chain,

$$f_{a_n} = \alpha f_{a_{n-1}} + \sum_{j \in I_{b_n}, j \neq a_n} \alpha_j f_j$$

for some $\alpha \neq 0$. Therefore, either $(u_1, v_1), \dots, (u_k, v_k), (a_n, b_n)$ is a chain or $a_n = u_i$ for some $i \le k$, either of which contradicts the minimality of $n$. □

## 3. Redundant versions of the Rado-Horn Theorem in the finite case

In this section, we will prove a generalization of the finite version of the Rado-Horn Theorem, which is where the main difficulty in proving the Rado-Horn Theorem lies. In the papers [1, 2], the extensions to countable sets are given by a version of Tychonoff's Theorem. A similar extension of a corollary to our main theorem will also be given for the countable case. As mentioned in the introduction, the key to our development is understanding the partition of the indexing set that maximizes the sums of the dimensions in the following technical lemma.

**Lemma 9.** *Let $\{f_i : i \in I\}$ be a countable collection of vectors, and $M \in \mathbb{N}$. There exists a partition $I_1, \dots, I_M$ of $I$ maximizing $\sum_{j=1}^{M} \dim \operatorname{span}(\mathbb{F}_{I_j})$ such that $\mathbb{F}_{I_2}, \dots, \mathbb{F}_{I_m}$ are linearly independent. Moreover, let $L = \{i \in I_1 : f_i = \sum_{j \in I_1, j \neq i} \alpha_j f_j\}$, $L_0 = \{i \in I :$ there is a chain starting in $L$ and ending at $i\}$, and $L_j = L_0 \cap I_j$ for $1 \le j \le M$. If $(a_1, b_1), \dots, (a_n, b_n)$ is a chain of minimal length starting in $L$ and ending at $a_n$, then $f_{a_n} \in \operatorname{span}(\mathbb{F}_{L_m})$ for all $1 \le m \le M$.*

*Proof.* The first part of the lemma follows immediately from Lemma 6. We show that, if $(a_1, b_1), \dots, (a_n, b_n)$ is a chain of minimal length starting in $L$ and ending at $a_n$, then $f_{a_n} \in \operatorname{span}(\mathbb{F}_{L_m})$ for each $1 \le m \le M$.

For $n = 1$, fix $m \in \{1, \ldots, M\}$, and observe that $a_1 \in L$. Hence, by Lemma 6, we can write $f_{a_1} = \sum_{l \in I_m} \alpha_l f_l$. For each $l$ such that $\alpha_l \neq 0$, $(a_1, 1), (l, m)$ is a chain ending at $l$. Therefore, $f_{a_1} \in \operatorname{span}(\mathbb{F}_{L_m})$, as desired.

Since $\mathbb{F}_{I_2}, \ldots, \mathbb{F}_{I_M}$ are linearly independent, $L = \{i \in I_1 : f_i = \sum_{j \in I_1, j \neq i} \alpha_j f_j\}$, and $(a_1, b_1), \ldots, (a_n, b_n)$ is a chain of minimal length, it follows that for each $1 \leq i < n$, $b_i \neq b_{i+1}$.

Therefore, proceeding by induction, we can define

$$U_k^1 = I_k, \quad 1 \leq k \leq M,$$

and for $2 \leq i \leq n$,

$$\begin{aligned} U_k^i &= U_k^{i-1} \text{ for } k \neq b_{i-1}, k \neq b_i, \\ U_{b_i}^i &= U_{b_i}^{i-1} \cup \{a_{i-1}\}, \\ U_{b_{i-1}}^i &= U_{b_{i-1}}^{i-1} \setminus \{a_{i-1}\}. \end{aligned}$$

**Claim 10.** *For each $1 \leq i \leq n$, $f_{a_i}$ can be written as the sum*

$$f_{a_i} = \sum_{j \in I_{b_i}, j \notin \{a_p : 1 \leq p \leq n\}} \alpha_j f_j + \sum_{j \in U_{b_i}^i \cap \{a_p : 1 \leq p < i\}} \alpha_j f_j. \tag{4}$$

*Proof of claim.* For the case $i = 1$, note that $a_1 \in L$ implies that $f_{a_1} = \sum_{j \in L, j \neq a_1} \alpha_j f_j$ for some choice of $\alpha_j$. By Lemma 8 none of these $j \in L$ can be in $\{a_p : 1 \leq p \leq n\}$ since this would not be a chain of minimal length. Recalling that $b_i = 1$, the claim is proven for $i = 1$.

Proceeding by induction, let $i \in \{1, \ldots, n\}$ and we assume (4) is true for $1 \leq k < i$. We will show that it is also true for $i$. Note that

$$\begin{aligned} f_{a_i} &= \alpha f_{a_{i-1}} + \sum_{j \in I_{b_i}, j \neq a_i} \alpha_j f_j & (5) \\ &= \alpha f_{a_{i-1}} + \sum_{j \in I_{b_i} \cap U_{b_i}^i, j \neq a_i} \alpha_j f_j + \sum_{j \in I_{b_i} \setminus U_{b_i}^i} \alpha_j f_j \\ &= \alpha f_{a_{i-1}} + \sum_{j \in I_{b_i} \cap U_{b_i}^i, j \neq a_i} \alpha_j f_j + \sum_{j \in I_{b_i} \cap \{a_p : 1 \leq p < i-1\}} \alpha_j f_j, & (6) \end{aligned}$$

where we have used in the last two lines that $I_{b_i} \cap \{a_p : 1 \leq p < i - 1\} = I_{b_i} \setminus U_{b_i}^i$. Now, suppose for the sake of contradiction that there is a $j \in I_{b_i} \cap U_{b_i}^i$ such that $\alpha_j \neq 0$ and $j = a_p$ for some $p > i$. Then, $(a_1, b_1), \ldots, (a_{i-1}, b_{i-1}), (a_p, b_i)$ is a chain, which contradicts the minimality of the chain $(a_1, b_1), \ldots, (a_n, b_n)$. So, using the induction hypothesis on the last term in (6) and combining terms, one obtains

$$f_{a_i} = \alpha f_{a_{i-1}} + \sum_{j \in I_{b_i}, j \notin \{a_p : 1 \leq p \leq n\}} \alpha_j f_j + \sum_{j \in U_{b_i}^i \cap \{a_p : 1 \leq p < i\}} \alpha_j f_j.$$

□

Continuing the proof of Lemma 9, by Claim 10 and the fact that $I_{b_i} \setminus \{a_p : 1 \leq p \leq n\} \subset U^k_{b_i}$ for all $1 \leq k \leq n$, we have that $f_{a_i} \in \mathrm{span}\,(\mathbb{F}_{U^i_{b_i} \setminus \{a_i\}})$. Therefore, $\dim\mathrm{span}\,(\mathbb{F}_{U^i_{b_i}}) = \dim\mathrm{span}\,(\mathbb{F}_{U^{i+1}_{b_i}})$. In particular,

$$\sum_{k=1}^{M} \dim\mathrm{span}\,(\mathbb{F}_{U^i_k}) = \sum_{k=1}^{M} \dim\mathrm{span}\,(\mathbb{F}_{I_k}) \tag{7}$$

is a maximum for each $i$.

We turn now to finishing the proof of the lemma; namely, we show that $f_{a_n} \in \mathrm{span}\,(\mathbb{F}_{L_m})$ for each $1 \leq m \leq M$. By (7), Claim 10, and Lemma 6, $f_{a_n} \in \mathrm{span}\,(\mathbb{F}_{U^n_m})$ for each $1 \leq m \leq M$. Therefore, for $m \neq b_n$, there exist $\alpha^0_j$ such that

$$\begin{aligned} f_{a_n} &= \sum_{j \in U^n_m} \alpha^0_j f_j = \sum_{j \in U^n_m \cap I_m} \alpha^0_j f_j + \sum_{j \in U^n_m \setminus I_m} \alpha^0_j f_j \\ &= \sum_{j \in U^n_m \cap I_m} \alpha^0_j f_j + \sum_{j \in \{a_p : b_{p+1} = m, 1 \leq p < n-1\}} \alpha^0_j f_j. \end{aligned} \tag{8}$$

By definition of a chain, for each $a_p$ such that $b_{p+1} = m$ and $1 \leq p < n-1$,

$$f_{a_p} = \alpha^p f_{a_{p+1}} + \sum_{j \in I_m, j \neq a_{p+1}} \alpha^p_j f_j, \tag{9}$$

for some choice of $\alpha^p_j$ and some $\alpha^p \neq 0$.

Fix $j_0$ such that $\alpha^0_{j_0} \neq 0$ in (8). We show that $j_0 \in L_m$, which finishes the proof of the lemma. Clearly, if $j_0 \in \{a_1, \ldots, a_n\}$, then we are done, so we assume that $j_0 \notin \{a_1, \ldots, a_n\}$.

Case 1: There is some $1 \leq p < n-1$ such that $b_{p+1} = m$ and $\alpha^p_{j_0} \neq 0$. Then, one can solve (9) for $f_{j_0}$ to obtain

$$f_{j_0} = \beta f_{a_p} + \sum_{j \in I_m, j \neq j_0, j \neq a_p} \beta_j f_j$$

for some $\beta \neq 0$. Hence, $(a_1, b_1), \ldots, (a_p, b_p), (j_0, m)$ is a chain and $j_0 \in L_m$.

Case 2: For each $1 \leq p < n-1$ such that $b_{p+1} = m$, we have $\alpha^p_{j_0} = 0$. We have

$$\begin{aligned} f_{a_n} &= \sum_{j \in U^n_m \cap I_m} \alpha^0_j f_j + \sum_{j \in \{a_p : b_{p+1} = m, 1 \leq p < n-1\}} \alpha^0_j f_j \\ &= \sum_{j \in U^n_m \cap I_m} \alpha^0_j f_j + \sum_{p \in \{p : b_{p+1} = m, 1 \leq p < n-1\}} \alpha^0_{a_p} f_{a_p} \\ &= \sum_{j \in U^n_m \cap I_m} \alpha^0_j f_j + \sum_{p \in \{p : b_{p+1} = m, 1 \leq p < n-1\}} \alpha^0_{a_p} \big(\alpha^p f_{a_{p+1}} + \sum_{j \in I_m, j \neq a_{p+1}} \alpha^p_j f_j\big) \\ &= \alpha^0_{j_0} f_{j_0} + \sum_{j \in I_m, j \neq j_0} \tilde{\alpha}_j f_j, \end{aligned}$$

where the first equality is (8), the second equality is a re-indexing, the third equality follows from (9), and the last equality holds for some choice of $\tilde{\alpha}_j$ by combining sums, since $\alpha^p_{j_0} = 0$ for all $1 \le p < n-1$ such that $b_{p+1} = m$, and $j_0 \notin \{a_1, \ldots, a_n\}$. Therefore, $(a_1, b_1), \ldots, (a_n, b_n), (j_0, m)$ is a chain and $j_0 \in L_m$. □

**Theorem 11.** *Let $\{f_i : i \in I\}$ be a finite collection of vectors in a vector space $X$ and $M \in \mathbb{N}$. Then the following conditions are equivalent.*

(i) *There exists a partition $\{I_j : j = 1, \ldots, M\}$ of $I$ such that for each $1 \le j \le M$ the set $\mathbb{F}_{I_j}$ is linearly independent.*

(ii) *For all $J \subset I$,*

$$\frac{|J|}{\dim \operatorname{span}(\mathbb{F}_J)} \le M. \tag{10}$$

*Moreover, in the case that either of the conditions above fails, there exists a partition $\{I_j : j = 1, \ldots, M\}$ of $I$ and a subspace $S$ of $X$ such that the following three conditions hold.*

(a) *For all $1 \le j \le M$, $S = \operatorname{span}\{f_i : i \in I_j \text{ and } f_i \in S\}$.*

(b) *For $J = \{i \in I : f_i \in S\}$, $\frac{|J|}{\dim \operatorname{span}(\mathbb{F}_J)} > M$.*

(c) *For each $1 \le j \le M$, $\sum_{i \in I_j, f_i \notin S} \alpha_i f_i \in S$ implies $\alpha_i = 0$ for all $i$. In particular, for each $1 \le j \le M$, $\{f_i : i \in I_j, f_i \notin S\}$ is linearly independent.*

*Proof.* We include a proof of the implication (i) ⇒ (ii) for completeness. Let $\{I_j : 1 \le j \le M\}$ be a partition of $I$ such that $\mathbb{F}_{I_j}$ is linearly independent for $1 \le j \le M$. Let $J \subset I$ and consider $J_j = I \cap I_j$, $1 \le j \le M$. Then,

$$|J| = \sum_{j=1}^{M} |J_j| = \sum_{j=1}^{M} \dim \operatorname{span}(\mathbb{F}_{J_j}) \le M \dim \operatorname{span}(\mathbb{F}_J),$$

as desired.

We prove (ii) ⇒ (i) and the moreover part together. Let $\{I_j : j = 1, \ldots, M\}$ be a partition of $I$ guaranteed to exist by Lemma 9. Suppose that this doesn't partition $\mathbb{F}_I$ into linearly independent sets, i.e. $\mathbb{F}_{I_1}$ is not linearly independent. As in Lemma 9, let $L = \{i \in I_1 : f_i = \sum_{j \in I_1, j \ne i} \alpha_j f_j\}$ be the index set of the "linearly dependent vectors"in $I_1$, $L_0 = \{i \in I : \text{there is a chain starting in } L \text{ ending at } i\}$, and $L_j = L_0 \cap I_j, 1 \le j \le M$.

Let $S = \operatorname{span}(\mathbb{F}_{L_0})$. By Lemma 9, $S = \operatorname{span}(\mathbb{F}_{L_j})$ for all $1 \le j \le M$. Moreover, for $1 \le j \le M$, $i \in L_j$ implies that $i \in I_j$ and $f_i \in S$. Therefore,

$$S \subset \operatorname{span}\{f_i : i \in L_j\} \subset \operatorname{span}\{f_i : i \in I_j, f_i \in S\} = S,$$

and (a) is proven.

To see (b), let $J = \{i \in I : f_i \in S\}$. By construction, $L \subset J$. Let $d = \dim(S)$ and see that, by (a), $\dim \operatorname{span}(\mathbb{F}_J) = d$. Moreover,

$$|J| = |L_1| + \cdots + |L_M| = |L_1| + (M-1)d > dM,$$

since $L_1$ is linearly dependent. Therefore, (b) is satisfied.

Finally, we show (c). Let $P_j = \{i \in I_j : f_i \notin S\}$, $Q_j = I_j \setminus P_j$. Suppose $g = \sum_{i \in P_j} \alpha_i f_i \in S$. By (a), $g$ can also be written as the linear combination $g = \sum_{i \in Q_j} \alpha_i f_i$, which implies that either $\alpha_i = 0$ for all $i \in P_j$ or there exists $k \in P_j$ such that $f_k = \sum_{i \in I_j, i \neq k} \alpha_i f_i$. Therefore, by our assumption that all linearly dependent vectors are in $I_1$ and by the definition of $L$, it follows that $k \in L$ and $f_k \in S$. This cannot be, so $\alpha_i = 0$ for all $i \in P_j$. □

In the following result, we prove a more direct generalization of the Rado-Horn Theorem in the finite case. One main ingredient for the proof is Theorem 11. (Added after publication: this theorem can also be proven using a generalized version of Rado-Horn to matroids found in [8].)

**Theorem 12.** *Let $I$ be a finite index set, $\{f_i : i \in I\}$ be a collection of vectors in a vector space, and $K, M \in \mathbb{N}$. Then the following conditions are equivalent.*

(i) *There exists a subset $H \subset I$ with $|H| = K$ such that $\{f_i : i \in I \setminus H\}$ can be written as the union of $M$ linearly independent sets*
(ii) *For every $J \subset I$,*

$$\frac{|J| - K}{\dim \operatorname{span}\left(\{f_i : i \in J\}\right)} \leq M. \tag{11}$$

*Proof.* For the implication (i) ⇒ (ii), if $J \subset I$, then

$$\frac{|J| - K}{\dim \operatorname{span}\left(\mathbb{F}_J\right)} \leq \frac{|J \cap (I \setminus H)|}{\dim \operatorname{span}\left(\mathbb{F}_{J \setminus H}\right)} \leq M,$$

by Theorem 1.

For the reverse direction, let $S$ and the partition $\{I_j : j = 1, \ldots, M\}$ be as in Theorem 11. For each $1 \leq j \leq M$, let $\tilde{I}_j$ be a minimal spanning set for $\mathbb{F}_{I_j}$. Let $H = I \setminus \cup_{j=1}^M \tilde{I}_j$. Clearly, $\{\tilde{I}_j : 1 \leq j \leq M\}$ is a partition of $I \setminus H = \cup_{j=1}^M \tilde{I}_j$ such that each $\mathbb{F}_{\tilde{I}_j}$ is linearly independent; it remains to show that $|H| = \sum_{j=1}^M |I_j \setminus \tilde{I}_j| \leq K$.

Let $P_j = \{i \in I_j : f_i \in S\}$ and $Q_j = \tilde{I}_j \setminus P_j$. To this end, we first claim that

$$I_j \setminus \tilde{I}_j \subset P_j \quad \text{for each } 1 \leq j \leq M. \tag{12}$$

For this, fix $1 \leq j \leq M$ and let $i \in I_j \setminus \tilde{I}_j$. Assume that $i \notin P_j$. Then, $f_i \notin S$ and $f_i \notin \tilde{I}_j$. Since $\mathbb{F}_{\tilde{I}_j}$ is a spanning set, $f_i \in \operatorname{span}\{f_k : k \in \tilde{I}_j\} \subset \operatorname{span}\{f_k : k \in I_j, k \neq i\}$. Therefore, we can write $f_i = \sum_{k \in I_j, k \neq i} \alpha_k f_k$ for some choice of $\alpha_k$. Grouping all of the terms not in $S$ with $f_i$ yields a contradiction to Theorem 11 (c). This proves (12).

Secondly, we will show that $\mathbb{F}_{P_j \cap \tilde{I}_j}$ is a basis for $S$. Indeed, let $f \in S$. Since the span of $\mathbb{F}_{\tilde{I}_j}$ contains $S$, we have that $f = g + h$, where $g \in \operatorname{span}(\mathbb{F}_{P_j \cap \tilde{I}_j})$ and $h \in \operatorname{span}(\mathbb{F}_{Q_j})$. By Theorem 11 (c) and the fact that $f, g \in S$, $h = 0$ and $f \in \operatorname{span}(\mathbb{F}_{P_j \cap \tilde{I}_j})$.

Employing (12), the fact that $\mathbb{F}_{P_j \cap \tilde{I}_j}$ is a basis for $S$, and (11) yields

$$\begin{aligned}\sum_{j=1}^{M} |I_j \setminus \tilde{I}_j| &= \sum_{j=1}^{M} |P_j \setminus \tilde{I}_j| \\ &= \sum_{j=1}^{M} |(P_j \setminus \tilde{I}_j) \cup (P_j \cap \tilde{I}_j)| - \sum_{j=1}^{M} |P_j \cap \tilde{I}_j| \\ &= \sum_{j=1}^{M} |P_j| - M \dim S \\ &= \left| \bigcup_{j=1}^{m} P_j \right| - M \dim \operatorname{span}(\mathbb{F}_{\cup P_j}) \leq K.\end{aligned}$$

This proves the theorem. ∎

## 4. Proof of Theorem 2

First, we will require the following technical lemma, which will be the main ingredient in the proof of Theorem 2.

**Lemma 13.** *Let* $\{f_i : i \in \mathbb{N}\}$ *be a collection of vectors in a vector space and let* $I_N = \{i \in \mathbb{N} : 1 \leq i \leq N\}$. *If there exists* $K \in \mathbb{N}$ *such that*

$$\frac{|J| - K}{\dim \operatorname{span}(\mathbb{F}_J)} \leq M \tag{13}$$

*for all finite* $J \subset \mathbb{N}$, *then there exists* $H \subset \mathbb{N}$ *such that* $|H| = K$ *and for all* $N \geq 1$, $\mathbb{F}_{I_N \setminus H}$ *can be written as the union of* $M$ *linearly independent sets.*

*Proof.* Choose the smallest $K$ such that (13) holds. Then, there exists a finite $J \subset \mathbb{N}$,

$$\frac{|J| - (K-1)}{\dim \operatorname{span}(\mathbb{F}_J)} > M. \tag{14}$$

Let $A$ be the largest element in $J$ and fix $N \geq A$. By Theorem 12 there exists $H_N \subset I_N$ such that $|H_N| \leq K$ and $\mathbb{F}_{I_N \setminus H_N}$ can be written as the union of $M$ linearly independent sets. By (14), $|H_N| = K$. We show that $H_N \subset I_A$. If not, then $\mathbb{F}_{I_A \setminus (H_N \cap I_A)}$ can be written as the union of $M$ linearly independent sets, but $|H_N \cap I_A| < K$, which together with equation (14) would contradict Theorem 12.

So, for every $N \geq A$, there exists $H_N \subset I_A$ such that $\mathbb{F}_{I_N \setminus H_N}$ can be written as the union of $M$ linearly independent sets. Since there are only finitely many subsets of $I_A$, there exist $N_1 < N_2 < N_3 < \cdots$ such that for all $i, j \in \mathbb{N}$ we have $H_{N_i} = H_{N_j}$. Write $H = H_{N_1}$. Then, for any $N$, there exist $N_i > N$ and $H = H_{N_i} \subset I_A \subset I_N$ such that $\mathbb{F}_{I_{N_i} \setminus H}$ can be written as the union of $M$ linearly independent sets. Therefore, $\mathbb{F}_{I_N \setminus H}$ can be written as the union of $M$ linearly independent sets. ∎

We finish by proving Theorem 2. As in [1, 2], we could extend Theorem 12 to the countable setting using a selection theorem. Easier in our case is to apply the infinite version of the Rado-Horn Theorem directly.

*Proof of Theorem 2.* By Lemma 13 and the implication $(i) \Rightarrow (ii)$ from the Rado-Horn Theorem, there is a single set $H$ such that $|H| = K$ and for every finite set $J \subset I \setminus H$,

$$\frac{|J|}{\dim \operatorname{span}(\mathbb{F}_J)} \leq M.$$

Thus, the hypotheses of the infinite version of the Rado-Horn Theorem are satisfied for $I \setminus H$, and $\mathbb{F}_{I \setminus H}$ can be written as the union of $M$ linearly independent sets. □

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