ON Σ-q RINGS

S. K. JAIN, SURJEET SINGH, AND ASHISH K. SRIVASTAVA

Abstract. Nakayama (Ann. of Math. 42, 1941) showed that over an artinian serial ring every module is a direct sum of uniserial modules. Hence artinian serial rings have the property that each right (left) ideal is a finite direct sum of quasi-injective right (left) ideals. A ring with the property that each right (left) ideal is a finite direct sum of quasi-injective right (left) ideals will be called a right (left) Σ-q ring. For example, commutative self-injective rings are Σ-q rings. In this paper, various classes of such rings that include local, simple, prime, right non-singular right artinian, and right serial, are studied. Prime right self-injective right Σ-q rings are shown to be simple artinian. Right artinian right non-singular right Σ-q rings are upper triangular block matrix rings over rings which are either zero rings or division rings. In general, Σ-q ring is not left-right symmetric nor is it Morita invariant.

1. Introduction

Artinian serial rings[1] introduced by Asano and Köthe is an important class of rings that appear at a number of places in the study of theory of rings and modules (see [5] and [18] for details). For example, these are rings of a finite representation type. Nakayama showed that every module over an artinian serial ring is a direct sum of uniserial modules [17]. Fuller proved that every indecomposable module over an artinian serial ring is quasi-injective [7]. Therefore, every right ideal in an artinian serial ring is a finite direct sum of quasi-injective right ideals. In this paper we study rings having the property that each right ideal is a finite direct sum of quasi-injective right ideals. Such rings will be called right *Σ-q rings*. In particular, rings in which each right ideal is quasi-injective were studied by Beidar et al. [2], Byrd [3], Hill [9], Ivanov ([10], [11]) and Jain et al. [12] and are known as *right q-rings*. Jain et al. [12] proved that a ring is a right q-ring if and only if it is right self-injective and each essential right ideal is two-sided. In Beidar et al. [2], among others, non-local indecomposable right q-rings were shown to be either semisimple artinian or of the form

$$\begin{bmatrix} D & V & 0 & . & . & 0 \\ 0 & D & V & 0 & . & . \\ . & . & D & V & 0 & . \\ . & . & . & D & V & 0 \\ . & . & . & . & D & V \\ V(\alpha) & 0 & . & . & . & D \end{bmatrix},$$ where V is one-dimensional both as a left D-space and a right D-space, $V(\alpha)$ is also a one-dimensional left D-space as well as a

2000 *Mathematics Subject Classification.* 16D50, 16D70, 16D80.
Key words and phrases. quasi-injective ideals, q-rings, self-injective rings, artinian serial rings.

[1]Generalized uniserial rings in the terminology of Nakayama

right D-space with right scalar multiplication twisted by an automorphism α of D. The purpose of this paper is to study right Σ-q rings and to provide a description of indecomposable right non-singular right artinian right Σ-q rings.

A right Σ-q ring need not be a right q-ring. For example, a matrix ring over an artinian serial ring, that is not semisimple, is a right Σ-q ring but not a right q-ring. We show that a simple right Σ-q ring must be an artinian ring. Lemma 5 shows that if $\mathbb{M}_n(R)$ is a right Σ-q ring then R is also a right Σ-q ring. One of our main results is that a prime right self-injective right Σ-q ring is simple artinian (Theorem 7). As a consequence it follows that the ring of all linear transformations on a vector space V is a right Σ-q ring if and only if V is finite dimensional. The notion of Σ-q ring is not left-right symmetric. We give an example of an incidence ring which is a left Σ-q ring but not a right Σ-q ring (Example 16). Also, we show that right (left) Σ-q ring is not Morita invariant. We conclude by giving the structure of a right artinian right non-singular right Σ-q ring as a triangular matrix ring of certain block matrices (Theorems 23-24).

2. Definitions and Notations

All rings considered in this paper have unity and all modules are right unital unless stated otherwise. A right R-module M is called quasi-injective if $Hom_R(-, M)$ is right exact on all short exact sequences of the form $0 \longrightarrow K \to M \longrightarrow L \longrightarrow 0$. Johnson and Wong [13] characterized quasi-injective modules as those that are fully invariant under endomorphisms of their injective hulls. A ring R is called von Neumann regular if every principal right (left) ideal of R is generated by an idempotent. A von Neumann regular ring is called abelian if all its idempotents are central. A ring R is called a right (left) duo ring if every right (left) ideal of R is two-sided. If a ring is both right duo and left duo then it is called a duo ring. It is known that a right self-injective right duo ring is a duo ring (see Remark 2.3, page 314, [2]). A ring R is called directly finite if $xy = 1$ implies $yx = 1$, for all $x, y \in R$. The index of a nilpotent element x in a ring R is the least positive integer n such that $x^n = 0$. The index of a two-sided ideal J in R is the supremum of the indices of all nilpotent elements of J. If this supremum is finite, then J is said to have bounded index. A module is called uniserial if its submodules are linearly ordered with respect to inclusion. A ring R is called a right (left) serial ring if R_R ($_RR$) is a direct sum of uniserial modules. If a ring is both left as well as right serial ring then it is called a serial ring. A ring R is called semiperfect if $R/J(R)$ is semisimple artinian and idempotents modulo $J(R)$ can be lifted. A right R-module M is called linearly compact in the discrete topology if any finite solvable system $\{x \equiv x_a(mod\ I_a) : a \in A\}$ of congruences is solvable for any index set A, where $x_a \in M$ and I_a is a submodule. A ring R is called right linearly compact ring if R_R is linearly compact. Any right linearly compact ring is semiperfect [19]. A right ideal I of R is called an essential right ideal if $I \cap J \neq 0$ for every non-zero right ideal J of R. We will denote by $Soc(M)$ and $E(M)$, respectively, the socle and injective hull of M. Two rings R and S are said to be Morita equivalent if there exists a category equivalence $F : mod\text{-}R \longrightarrow mod\text{-}S$. A ring theoretic property $\mathcal{P}$ is said to be Morita invariant if, whenever R has the property $\mathcal{P}$, so does every ring Morita equivalent to R. Let X be a finite partially ordered set and R any ring. The incidence ring of X with coefficients in R, denoted by $I(X, R)$, is the ring of functions $\{f : X \times X \longrightarrow R$ such that $f(x, y) = 0$ for each $x \nleq y\}$; multiplication is given by $(fg)(x, y) = \sum_{x \leq z \leq y} f(x, z)g(z, y)$. The ring $I(X, R)$ may also be viewed as the ring of square matrices with entries in R, whose rows and columns are indexed by X, where the (x, y) entry is 0 whenever $x \nleq y$.

3. Preliminaries

We will start with a basic lemma.

Lemma 1. *Let R be a right self-injective right Σ-q ring then each essential right ideal E of R is of the form $E = e_1T_1 \oplus e_2T_2 \oplus ... \oplus e_nT_n$ where $T_1, ..., T_n$ are two-sided ideals of R and $e_1, ..., e_n$ are orthogonal idempotents with $e_1 + e_2 + ... + e_n = 1$.*

Proof. By hypothesis, $E = E_1 \oplus E_2 \oplus ... \oplus E_n$, where each E_i is a quasi-injective right ideal. This gives $R = \widehat{E}_1 \oplus ... \oplus \widehat{E}_n$, where $\widehat{E}_i$ denotes injective hull of E_i. Write each $\widehat{E}_i$ as e_iR with $e_1 + e_2 + ... + e_n = 1$. Since each E_i is quasi-injective, we have $e_iR\ e_iE_i = E_i$. This gives $e_iRE_i = E_i$. Denote each two-sided ideal RE_i as T_i. So, we have $E = e_1T_1 \oplus e_2T_2 \oplus ... \oplus e_nT_n$ where $T_1, ..., T_n$ are two-sided ideals of R and $e_1, ..., e_n$ are orthogonal idempotents with $e_1 + e_2 + ... + e_n = 1$. □

Lemma 2. *Let R be a right Σ-q ring with no nontrivial idempotents. Then R is a right q-ring and hence a duo ring.*

Proof. If R is a right Σ-q ring with no nontrivial idempotents then R is right self-injective and by Lemma 1, each essential right ideal of R is two-sided. Hence R is a right q-ring [12]. Since R has no nontrivial idempotents, it must be right duo ([12], Theorem 2.3). Now, since a right self-injective right duo ring must be a duo ring (see [2], page 314, Remark 2.3), R is a duo ring. □

Corollary 3. *Let R be a local ring. Then R is a right Σ-q ring if and only if R is a right q-ring if and only if R is a right self-injective duo ring.*

Proof. This follows from the above lemma. □

Proposition 4. *If R is a right Σ-q ring and e is an idempotent in R such that $ReR = R$ then eRe is also a right Σ-q ring.*

Proof. We know that if $ReR = R$, then *mod-R* and *mod-eRe* are Morita equivalent under the functors given by $\mathcal{F} : \textit{mod-}R \longrightarrow \textit{mod-}eRe$, $\mathcal{G} : \textit{mod-}eRe \longrightarrow \textit{mod-}R$ such that for any M_R, $\mathcal{F}(M) = Me$ and for any module T over eRe, $\mathcal{G}(T) = T \otimes_{eRe} eR$.

Suppose R is a right Σ-q ring. Let A be any right ideal of eRe. Then $AeR \cong A \otimes_{eRe} eR$ and AeR is a right ideal of R. Therefore, $AeR = A_1 \oplus ... \oplus A_n$ where A_i's are quasi-injective right ideals in R. By Morita equivalence we get that each A_ie is quasi-injective as an eRe-module. Then $A = AeRe = A_1e \oplus ... \oplus A_ne$ is a direct sum of quasi-injective right ideals. Hence eRe is a right Σ-q ring. □

Lemma 5. *If $\mathbb{M}_n(R)$ is a right Σ-q ring, then R is also a right Σ-q ring.*

Proof. We have $R \cong e_{11}\mathbb{M}_n(R)e_{11}$ and $\mathbb{M}_n(R)e_{11}\mathbb{M}_n(R) = \mathbb{M}_n(R)$, where e_{11} is the usual matrix unit. Therefore, the result follows from the above proposition. □

Later, in Example 18, we will show that if R is a right Σ-q ring then $\mathbb{M}_n(R)$ need not be a right Σ-q ring.

4. Prime Rings, Right Self-injective Rings, and Regular Rings

First, we consider a prime right self-injective right Σ-q ring.

Lemma 6. *A prime right self-injective right Σ-q ring must be von Neumann regular.*

Proof. We prove $Z(R_R) = 0$. If possible, let x be a non-zero element in $Z(R_R)$. Then there exists an essential right ideal E of R such that $xE = 0$. Now, by Lemma 1, $E = e_1T_1 \oplus e_2T_2 \oplus ... \oplus e_nT_n$ where $T_1, ..., T_n$ are two-sided ideals of R and $e_1, ..., e_n$ are orthogonal idempotents with $e_1 + e_2 + ... + e_n = 1$. Thus, $xe_iT_i = 0$. But since R is prime, $xe_i = 0$, for all i. Therefore, $x = 0$, a contradiction. Hence R is right non-singular and therefore, R must be von Neumann regular (see [15], page 362, Corollary 13.2). □

Theorem 7. *A prime right self-injective ring R is right Σ-q ring if and only if R is artinian.*

Proof. Let R be a prime right self-injective right Σ-q ring. By Lemma 6, R is von Neumann regular. Since the two-sided ideals of R are well-ordered (see [8], Proposition 8.5), R has a unique maximal two-sided ideal, say A. We claim R/A is simple artinian. If each maximal right ideal of R/A is a summand then each right ideal of R/A is a summand and hence R/A is simple artinian. Else, let N/A be a maximal essential right ideal of R/A. Then N is an essential maximal right ideal of R. By Lemma 1, $N = e_1T_1 \oplus e_2T_2 \oplus ... \oplus e_nT_n$, where $T_1, ..., T_n$ are two-sided ideals of R and $e_1, ..., e_n$ are orthogonal idempotents with $e_1 + e_2 + ... + e_n = 1$. Since R/N is a simple module, we have $\frac{R}{N} \cong \frac{e_iR}{e_iT_i}$. Now, since $\frac{e_iR}{e_iT_i}$ is a direct summand of $\frac{R}{T_i}$, it is projective as R/T_i-module. As R/N is an R/T_i-module and $T_i \subset A \subset N$, it follows that R/N is a simple, projective R/A-module. Therefore, R/A is a simple ring with non-zero socle and hence artinian. Thus, R has bounded index of nilpotence (see [8], page 79) and so R is artinian (see [8], Theorem 7.9).

The converse is obvious. □

We remark that if in the above theorem R is not a prime ring then it needs not be artinian. The following is an example of a non-prime von Neumann regular right self-injective right Σ-q ring which is not artinian.

Example 8. *Let S be an infinite boolean ring and suppose $R = Q^r_{\max}(S)$. Clearly R is commutative, von Neumann regular, self-injective and hence a Σ-q ring. But, R is not artinian.*

The following result is a consequence of Theorem 7.

Corollary 9. *The ring of linear transformations, $R = End_D(V)$ of a vector space V over a division ring D is a right Σ-q ring if and only if the vector space V is finite-dimensional .*

Next, we have the following

Corollary 10. *A simple ring R is a right Σ-q ring if and only if R is artinian.*

Proof. Let R be a simple right Σ-q ring. Then R contains a quasi-injective right ideal, say A. As R is a simple ring, $RA = R$. So, we have $R = \Sigma r_i A$ where $r_i \in R$. As $1 \in \Sigma_{i=1}^{k} r_i A$, we have $R = \Sigma_{i=1}^{k} r_i A$. Therefore, $R \cong A^k/L \cong S$ where S is a direct summand of A^k and L is a submodule of A^k. Because A is quasi-injective, A^k is quasi-injective. Therefore, S_R and hence R_R is quasi-injective. Thus, R is right self-injective. Now, by Theorem 7, R is artinian.

The converse is trivial. □

Remark 11. *The above proof shows, in particular, that a simple ring with a quasi-injective right ideal must be right self-injective. Dinh van Huynh has told us in a private communication that in his forthcoming joint paper with John Clark, they have obtained the same result.*

Next, we give an example of a simple right self-injective ring which is not a right Σ-q ring.

Example 12. *Let S be an integral domain which is not a right Ore domain. Consider $R = Q^r_{\max}(S)$. We know that R is a simple, right self-injective, von Neumann regular ring (see [15], page 377, Corollary $(13.38)'$). If R is a right Σ-q ring then by Corollary 10, R is artinian and hence S is right Ore, which is not true. Therefore, R is not a right Σ-q ring.*

Lemma 13. *Let R be a von Neumann regular ring. Suppose every ring homomorphic image of R is a right self-injective right Σ-q ring. Then R is semisimple artinian.*

Proof. Let P be a prime ideal of R. Then R/P is a prime von Neumann regular right self-injective right Σ-q ring. By Theorem 7, R/P is artinian. Since R is a von Neumann regular right self-injective ring and each primitive factor ring of R is artinian, we have $R \cong \Pi_{i=1}^{k} \mathbb{M}_{n_i}(S_i)$ where each S_i is an abelian regular right self-injective ring (see [8], Theorem 7.20). Thus, each S_i is a right self-injective duo ring.

Now, let C_i be any right ideal of S_i. Since S_i is a duo ring, C_i is a two-sided ideal and we have $\frac{\mathbb{M}_{n_i}(S_i)}{\mathbb{M}_{n_i}(C_i)} \cong \mathbb{M}_{n_i}(S_i/C_i)$. Since each $\mathbb{M}_{n_i}(S_i)$ satisfies the property that homomorphic images are right self-injective, $\mathbb{M}_{n_i}(S_i/C_i)$ is right self-injective and hence S_i/C_i is right self-injective. So, each cyclic right S_i- module is quasi-injective. Hence by Koehler [14], S_i is a finite direct sum of rings each of which is semisimple artinian or a rank 0 duo right linearly compact ring. But, since S_i is a von Neumann regular ring, we conclude that it must be semisimple artinian. Hence R is semisimple artinian. □

Next, we show

Proposition 14. *Let R be a right Σ-q ring and suppose every ring homomorphic image of $R/J(R)$ is a right self-injective right Σ-q ring, then R is semiperfect.*

Proof. By Lemma 13, $R/J(R)$ is semisimple artinian. Now, let the composition length of $R/J(R)$ be n. Then R_R cannot be a direct sum of more than n submodules. Thus, R_R is a finite direct sum of indecomposable right ideals, each of which is quasi-injective. Let $A = eR$ be any indecomposable summand of R_R. As $A = eR$

is quasi-injective, its ring of endomorphisms eRe is a local ring (see [15], page 244, ex. 6.32). Hence by ([1], Theorem 27.6), R is semiperfect. □

Example 18 shows, among others, that a right self-injective semiperfect ring need not be a right Σ-q ring.

For right self-injective right Σ-q rings, we ask the following question;

Problem 15. *Is every right self-injective right Σ-q ring a directly finite ring?*

One can answer this in the affirmative for right q-rings, but we are unable to answer this, in general.

5. Examples

Clearly, each right q-ring is a right Σ-q ring. However, there are plenty of examples of right Σ-q rings that are not right q-rings.

Recall from the introduction, that an artinian serial ring is a right Σ-q ring but such rings need not be q-rings. Furthermore, a left Σ-q ring need not be a right Σ-q ring. We give an example of an incidence ring which is a left Σ-q ring but not a right Σ-q ring.

Example 16. *Let $X = \{1, 2, 3, 4\}$ be a partially ordered set with $1 < 2 < 3$ and $1 < 2 < 4$. Let F be a field. The incidence ring is given as*

$$R = I(X, F) = \begin{bmatrix} F & F & F & F \\ 0 & F & F & F \\ 0 & 0 & F & 0 \\ 0 & 0 & 0 & F \end{bmatrix}.$$

R is a left artinian, left serial ring. Note that $Soc(_RR) = Fe_{11} + Fe_{12} + Fe_{13} + Fe_{14}$. It is a folklore that $_RR$ is non-singular. But for completeness, we prove this result. Suppose $Z(_RR) \neq 0$. Then there exists $z\ (\neq 0) \in soc(Z(_RR))$. Clearly, $z = a_{11}e_{11} + a_{12}e_{12} + a_{13}e_{13} + a_{14}e_{14}$. Then $e_{11}z = z$, which implies $e_{11} \notin l.ann(z)$ and therefore, $Fe_{11} \cap l.ann(z) = 0$. Hence, $l.ann(z)$ is not essential in $_RR$, a contradiction. So, $_RR$ is non-singular.

Re_{11} is simple, so quasi-injective. As Re_{11} is non-singular and quasi-injective, $End(Re_{11}) \cong End(E(Re_{11}))$ (see [15], page 272, ex. 7.32). Hence any endomorphism of $E = E(Re_{11})$ is given as multiplication by some element of F. Next, we show that Re_{22}, Re_{33}, and Re_{44} are quasi-injective.

The ring R is left serial, so Re_{33} is uniserial and hence uniform. We have $Soc(Re_{33}) = Fe_{13} \cong Fe_{11}$ as left R-module under the mapping $ae_{13} \longrightarrow ae_{11}$. Hence Re_{33}, being an essential extension of Fe_{13}, embeds in $E = E(Re_{11})$.

So, $\sigma(Re_{33}) \subseteq Re_{33}$, $\forall \sigma \in End(E)$. Therefore, Re_{33} is quasi-injective. Similarly, Re_{44} is quasi-injective. For every left ideal $C \subseteq Re_{33}$ and $\forall \sigma \in End(E)$, we have $\sigma(C) \subseteq C$ as Re_{33} is uniserial. Hence C is quasi-injective. Now, $Re_{22} \cong Re_{23} \subseteq Re_{33}$. Therefore, Re_{22} is quasi-injective. So, $_RR$ is a direct sum of quasi-injectives.

Next, we show that any indecomposable left ideal is uniserial and quasi-injective. Suppose there exists a left ideal $A \neq 0$ which is indecomposable and not uniform. Choose a left ideal A of smallest composition length. Let $\pi_i :_R R \longrightarrow Re_{ii}$ be *projection. If for some i, $\pi_i(A) = Re_{ii}$, we get $A = A' \oplus B'$ for some B', $A' \cong Re_{ii}$. This gives $B' = 0$ and $A \cong Re_{ii}$, a contradiction as A is not uniform. Therefore, $\pi_i(A) \subseteq J(R)e_{ii}$ for all i, which gives* $A \subseteq \begin{bmatrix} 0 & F & F & F \\ 0 & 0 & F & F \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B_2 \oplus B_3 \oplus B_4$.

Let $\pi_i' : J(R) \longrightarrow B_i$. Suppose $\pi_2'(A) \neq 0$. Now, B_2 is minimal and $B_2 \cong Re_{11}$ which is projective. So, $A \cong B_2$ which is uniform, a contradiction. Hence $\pi_2'(A) = 0$. This gives that $A \subseteq \begin{bmatrix} 0 & 0 & F & F \\ 0 & 0 & F & F \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = L$.

Clearly, $l.ann(L) = \begin{bmatrix} 0 & 0 & F & F \\ 0 & 0 & F & F \\ 0 & 0 & F & 0 \\ 0 & 0 & 0 & F \end{bmatrix}$

Let us denote $l.ann(L)$ by U, then L is a left R/U-module.

$R/U \cong \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$, *which is artinian serial.*

As $A \subseteq L$, we note that A is a module over $\begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$. *This implies that A is uniserial and hence uniform. Let B be any indecomposable left ideal in R then B is uniform. For some i, $\pi_i(soc(B)) \neq 0$ and so π_i is one-one on $soc(B)$. As B is uniform, $\pi_i|_B : B \longrightarrow Re_{ii}$ is one-one. We have $B \cong \pi_i(B) \subseteq Re_{ii}$. This gives that B is uniserial and quasi-injective. Now, let I be any left ideal of R. Then I is a finite direct sum of indecomposable left ideals. Since we have already proved above that any indecomposable left ideal is quasi-injective, I is a finite direct sum of quasi-injective left ideals. Thus, R is a left Σ-q ring.*

Now, we will show that R is not a right Σ-q ring. We have $e_{11}R = Fe_{11} + Fe_{12} + Fe_{13} + Fe_{14}$. Note that Fe_{13}, Fe_{14} are minimal right ideals. Clearly, $e_{11}R$ is not uniform and hence not quasi-injective. Therefore, R is not a right Σ-q ring.

We give below an example analogous to the above example that might be of interest to the reader.

Example 17. *Let $X = \{1, 2, 3, 4\}$ be a partially ordered set with $1 < 3 < 4$ and $2 < 3 < 4$. Let F be a field. Then the incidence ring is given as*

$$R = I(X, F) = \begin{bmatrix} F & 0 & F & F \\ 0 & F & F & F \\ 0 & 0 & F & F \\ 0 & 0 & 0 & F \end{bmatrix}.$$

By similar arguments as above R is a right Σ-q ring, but R is not a left Σ-q ring.

Let R be any right Σ-q ring and $S = \mathbb{M}_n(R)$. Now $R \cong e_{11}\mathbb{M}_n(R)e_{11}$ is a right Σ-q ring under the equivalence functor $\mathcal{F} : mod\text{-}\mathbb{M}_n(R) \longrightarrow mod\text{-}e_{11}\mathbb{M}_n(R)e_{11}$, where $\mathcal{F}(M) = Me_{11}$. In particular, $\mathcal{F}(e_{11}\mathbb{M}_n(R)) = e_{11}\mathbb{M}_n(R)e_{11}$ gives that any right ideal contained in $e_{11}\mathbb{M}_n(R)$ is a direct sum of finitely many quasi-injective modules. However, the ring $\mathbb{M}_n(R)$ need not be a right Σ-q ring.

The example which follows shows that matrix ring over a right Σ-q ring (in fact, even a q-ring) need not be a right Σ-q ring. Therefore, the notion of right Σ-q ring is not a Morita invariant property.

Example 18. *Let F be any field. Consider the ring $R = F[x, y]$ where $x^2 = 0$ and $y^2 = 0$. Then $R = F + Fx + Fy + Fxy$ is a commutative, local, artinian ring.*

Here, $J(R) = Fx + Fy + Fxy$.

Let $z = a + bx + cy + dxy \in ann(J)$. This implies that $(a + bx + cy + dxy)x = 0$, which gives $ax + cxy = 0$, and so, $a = 0$ and $c = 0$. Similarly, $(a+bx+cy+dxy)y = 0$, gives $a = 0$ and $b = 0$. So, we have $z = dxy$. Therefore, $soc(R) = Fxy$. Thus, R is a commutative local artinian ring with simple socle and hence a self-injective ring ([6], Page 217, Exercise 5). Since a commutative self-injective ring must be a q-ring, R is a q-ring.

Let $K = soc(R)$. Then R/K is of length 3 and $J(R/K) = A \oplus B$ where A and B are simple.

Let $T = \mathbb{M}_2(R)$. Then $T = e_{11}T \oplus e_{22}T$, where $e_{11}T \cong e_{22}T$.

Note $e_{11}M_2(R/K)/e_{11}M_2(J(R/K))$ is simple. Therefore, $e_{11}M_2(R/K)$ is local as $M_2(R/K)$ ($\cong \frac{M_2(R)}{M_2(K)}$)-module, and hence as $T = M_2(R)$-module. Observe that $e_{11}M_2(R/K)$ has two minimal right ideals $e_{11}A+e_{12}A$ and $e_{11}B+e_{12}B$. By suitably factoring, we then obtain a local T-module, say, N_T with $length(soc(N)) = 2$. If S_T is a simple module, then N is embeddable in $E(N) \cong E(S) \oplus E(S) \cong T_T$. Thus, N is a local, indecomposable, non-uniform module embeddable in T and so T contains a right ideal which is not quasi-injective. Therefore, T is not a right Σ-q ring.

6. Right non-singular right artinian rings, and right serial rings:

The lemma that follows is well-known.

Lemma 19. *(see [15], page 359, Theorem 13.1) Let R be right artinian and let eR be a right non-singular indecomposable quasi-injective right ideal, where e is an idempotent in R. Then eRe is a division ring.*

Lemma 20. *Let R be a right artinian right non-singular right Σ-q ring. Let e, f be any two indecomposable idempotents in R such that $eRf \neq 0$. Let $D = eRe$ and $D' = fRf$.*

(i) Then eRf is a one-dimensional left vector space over D.

(ii) For any $0 \neq z \in eRf$, there exists embedding $\sigma : D' \longrightarrow D$ such that for $fbf \in D'$, $zfbf = \sigma(fbf)z$.

(iii) If R is also right serial, then σ is an isomorphism.

Proof. (i) Consider any two non-zero elements erf and esf in eRf. Define a map $\phi : fR \longrightarrow erfR$ where $\phi(x) = erx$, for any $x \in fR$. This is clearly a well-defined surjective right R-homomorphism. Thus, we have $erfR \cong fR/Ker(\phi)$. Note that fR is indecomposable quasi-injective and hence uniform. If $Ker(\phi) \neq 0$ then as $Ker(\phi)$ is essential in fR, $fR/Ker(\phi)$ is singular. But, $erfR$ is non-singular, which gives a contradiction. Therefore, $Ker(\phi) = 0$ and hence $erfR \cong fR$. Similarly $esfR \cong fR$. Thus, we get an R-isomorphism $\sigma : erfR \longrightarrow esfR$ such that

$\sigma(erf) = esf$. Since eR is quasi-injective, we extend σ to $\eta : eR \longrightarrow eR$. Now $\eta(e) = eue$. Then $\sigma(erf) = euerf$. Therefore, $esf = euerf$. Hence eRf is one-dimensional left vector space over D.

(ii) Now $eRf = Dz$,for some $z \in eRf$. Therefore, given any $fbf \in fRf$, $zfbf = uz$, for some $u \in D$. This defines a monomorphism $\sigma : D^{'} \longrightarrow D$ such that $\sigma(fbf) = u$. Also, we have $\sigma(fbf)z = uz = zfbf$.

(iii) Consider any two non-zero elements erf and esf in eRf. As eR is uniserial, we may suppose $esfR \subseteq erfR$. Then $esf = erfuf$ for some $fuf \in fRf$. Since fRf is a division ring, it follows that $esfRf = erfRf$. Hence eRf is one-dimensional over $D^{'}$. Therefore, $eRf = Dz = zD^{'}$. From which it is immediate that σ is an isomorphism. □

Proposition 21. *Let R be a right artinian right non-singular right Σ-q ring.*

(i) If e, f are two indecomposable idempotents in R such that $eRf \neq 0$, then for any $0 \neq z \in eRf$, $eRez = zfRf$.

(ii) If R is an indecomposable ring, then $eRe \cong fRf$ for any two indecomposable idempotents e and f.

Proof. (i) As eR is quasi-injective and indecomposable, it is uniform. So, $Soc(eR)$ is simple. Let $Soc(eR) \cong R/I$ where I is a maximal right ideal of R. If I is essential in R, then R/I is singular, which is a contradiction. Therefore, I is a summand of R. Hence, $Soc(eR) \cong gR$ for some indecomposable idempotent g in R. Let $h : gR \longrightarrow Soc(eR)$ be an isomorphism. Since $h(g) \in eR$, we have $h(g) = ewg$ for some $w = ewg$. Hence $Soc(eR) = ewgR$. Let $\mu : gR \longrightarrow wgR$ be a non-zero R-homomorphism given by $\mu(gr) = wgr$. It is monic because gR is uniform. Then $gR \cong wgR$. Thus, $Soc(eR) = eRgR$. By Lemma 20, $eRew = eRg$. This yields, $Soc(eR) = eRewR = eRgR$. Consider any non-zero erg, $esg \in eRg$. As $Soc(eR)$ is simple, we get $eRgR = ergR = esgR$, and so $eRgRg = ergRg = esgRg$, whence it is easy to verify that, $eRg = eRew = wgRg$ for any non-zero $w \in eRg$. So the induced monomorphism $\eta : gRg \longrightarrow eRe$ given by $\eta(grg)w = wgrg$ is an isomorphism.

Consider an embedding $\lambda : fR \longrightarrow eR$, where $\lambda(fr) = zfr$. Then $\lambda(Soc(fR)) = zfRgR$. Let $v \in fRg$ be such that $zv \neq 0$. Then $w = zv \in eRg$. Now, v and w induce isomorphisms $\sigma_1 : gRg \longrightarrow fRf$, and $\sigma_2 : gRg \longrightarrow eRe$ respectively. Furthermore, $\sigma : fRf \longrightarrow eRe$ is a monomorphism induced by z. Because $\sigma_2 = \sigma\sigma_1$, σ is also an isomorphism.

(ii) Let $S = \{e_1, ..., e_m\}$ be a basic orthogonal set of indecomposable idempotents in R. For any two distinct members $e, f \in S$, set $e \leq f$ if $eRf \neq 0$, equivalently, if fR embeds in eR. This is a partial ordering on S. As R is indecomposable, S is connected. We may take $e, f \in S$. There exists a path $e = e_1, e_2, ..., e_k = f$. By definition, for any $i < k$, $e_iRe_{i+1} \neq 0$. By (i), $e_iRe_i \cong e_{i+1}Re_{i+1}$. Hence, $eRe \cong fRf$. □

If we assume that R is right serial in addition to being right artinian and right non-singular, we have the following equivalence. First recall that by Warfield [20], such a ring is right hereditary.

Theorem 22. *Let R be a right artinian right non-singular right serial ring. Then the following are equivalent;*

(i) R is a right Σ-q ring.

(ii) For any two indecomposable idempotents $e, f \in R$, if $eRf \neq 0$ then eRf is one-dimensional left vector space over eRe and one-dimensional right vector space over fRf.

Proof. (i)$\Longrightarrow$(ii) follows by Proposition 21.

Conversely, suppose (ii) holds. Let e be any indecomposable idempotent in R. Let A be a non-zero right ideal contained in eR. As R is right non-singular right artinian and right serial, A is projective and indecomposable. Therefore, $A \cong fR$ for some indecomposable idempotent $f \in R$. Let $\sigma : A \longrightarrow eR$ be a non-zero R-homomorphism. Then σ is a monomorphism and hence $A \cong \sigma(A)$. Because eR is uniserial, we have $\sigma(A) \subseteq A$. Since $Soc(eR)$ is simple, there exists an indecomposable idempotent $g \in R$ such that $Soc(eR) = eugR$ for some $u \in R$ and $\sigma(eug) = eugvg$ for some $v \in R$. As $eRg \neq 0$, by (ii), eRg is one-dimensional over eRe. Therefore, $eugvg = eweug$, for some non-zero ewe. Let η be the endomorphism of eR given by left multiplication by ewe. If $\lambda = \eta|_A$, then $\sigma - \lambda$ being zero on $Soc(eR)$, is zero. Hence η extends σ. So, eR is quasi-injective. This also proves that A is quasi-injective. In a right artinian right non-singular right serial ring, any right ideal is a finite direct sum of uniserial right ideals. As shown above any uniserial right ideal is quasi-injective. Hence R is a right Σ-q ring. □

Theorem 23. *Let R be an indecomposable right artinian right non-singular right Σ-q ring. Then*

$$R \cong \begin{bmatrix} \mathbb{M}_{n_1}(e_1Re_1) & \mathbb{M}_{n_1\times n_2}(e_1Re_2) & . & . & . & \mathbb{M}_{n_1\times n_k}(e_1Re_k) \\ 0 & \mathbb{M}_{n_2}(e_2Re_2) & . & . & . & \mathbb{M}_{n_2\times n_k}(e_2Re_k) \\ 0 & 0 & \mathbb{M}_{n_3}(e_3Re_3) & . & . & \mathbb{M}_{n_3\times n_k}(e_3Re_k) \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & \mathbb{M}_{n_k}(e_kRe_k) \end{bmatrix}$$

where e_iRe_i is a division ring, $e_iRe_i \cong e_jRe_j$ for each $1 \leq i, j \leq k$ and $n_1, ..., n_k$ are any positive integers. Furthermore, if $e_iRe_j \neq 0$, then it is one-dimensional left vector space over e_iRe_i and one-dimensional right vector space over e_jRe_j.

Proof. Let R be an indecomposable right artinian right non-singular right Σ-q ring. There exists an independent family $\mathcal{F} = \{e_iR : 1 \leq i \leq n\}$ of indecomposable right ideals such that $R = \oplus_{i=1}^{n} e_iR$. After renumbering, we may write $R = [e_1R] \oplus [e_2R] \oplus ... \oplus [e_kR]$, where for $1 \leq i \leq k$, $[e_iR]$ denotes the direct sum of those e_jR that are isomorphic to e_iR. Let $[e_iR]$ be a direct sum of n_i copies of e_iR. Consider $1 \leq i < j \leq k$. We arrange in such a way that $length(e_jR) \leq length(e_iR)$. Suppose $e_jRe_i \neq 0$, then we have an embedding of e_iR into e_jR, hence $length(e_iR) \leq length(e_jR)$. But by assumption $length(e_jR) \leq length(e_iR)$, so $length(e_iR) = length(e_jR)$, we get $e_jR \cong e_iR$, which is a contradiction. Hence $e_jRe_i = 0$ for $j > i$.

Thus, we have

$$R \cong \begin{bmatrix} \mathbb{M}_{n_1}(e_1Re_1) & \mathbb{M}_{n_1\times n_2}(e_1Re_2) & . & . & . & \mathbb{M}_{n_1\times n_k}(e_1Re_k) \\ 0 & \mathbb{M}_{n_2}(e_2Re_2) & . & . & . & \mathbb{M}_{n_2\times n_k}(e_2Re_k) \\ 0 & 0 & \mathbb{M}_{n_3}(e_3Re_3) & . & . & \mathbb{M}_{n_3\times n_k}(e_3Re_k) \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & \mathbb{M}_{n_k}(e_kRe_k) \end{bmatrix}$$

We have already seen earlier that each e_iRe_i is a division ring, $e_iRe_i \cong e_jRe_j$ for each $1 \leq i, j \leq k$ and if $e_iRe_j \neq 0$ then it is a one-dimensional left vector space over e_iRe_i as well as a one-dimensional right vector space over e_jRe_j (see Lemma 19 and Proposition 21). □

Let us consider the following condition*:*

(*) : For $1 \leq i, j \leq k$ with $i \neq j$ and primitive orthogonal idempotents e_i, e_j, either $e_iRe_j \neq 0$ or $e_jRe_i \neq 0$. In other words, for a right non-singular ring either e_iR is embeddable in e_jR or e_jR is embeddable in e_iR.

We remark that (*) holds if R is an indecomposable right non-singular serial ring.

Under the hypothesis (*) we have the following

Theorem 24. *Let R be an indecomposable right artinian right non-singular ring with the condition (*). Then R is a right Σ-q ring if and only if*

$$R \cong \begin{bmatrix} \mathbb{M}_{n_1}(D) & \mathbb{M}_{n_1\times n_2}(D) & . & . & . & \mathbb{M}_{n_1\times n_k}(D) \\ 0 & \mathbb{M}_{n_2}(D) & . & . & . & \mathbb{M}_{n_2\times n_k}(D) \\ 0 & 0 & \mathbb{M}_{n_3}(D) & . & . & \mathbb{M}_{n_3\times n_k}(D) \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & \mathbb{M}_{n_k}(D) \end{bmatrix}$$

where D is a division ring and $n_1, ..., n_k$ are any positive integers.

Proof. Let R be an indecomposable right artinian right non-singular right Σ-q ring. By the above theorem,

$$R \cong \begin{bmatrix} \mathbb{M}_{n_1}(e_1Re_1) & \mathbb{M}_{n_1\times n_2}(e_1Re_2) & . & . & . & \mathbb{M}_{n_1\times n_k}(e_1Re_k) \\ 0 & \mathbb{M}_{n_2}(e_2Re_2) & . & . & . & \mathbb{M}_{n_2\times n_k}(e_2Re_k) \\ 0 & 0 & \mathbb{M}_{n_3}(e_3Re_3) & . & . & \mathbb{M}_{n_3\times n_k}(e_3Re_k) \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & \mathbb{M}_{n_k}(e_kRe_k) \end{bmatrix}$$

where each e_iRe_i is a division ring and $e_iRe_i \cong e_jRe_j$ for each $1 \leq i, j \leq k$. Furthermore, by condition (*), we have $e_iRe_j \neq 0$ for each $1 \leq i < j \leq k$. Therefore, each e_iRe_j is a one-dimensional left vector space over e_iRe_i as well as a one-dimensional right vector space over e_jRe_j. Let us denote division ring e_iRe_i by D. Then we have

$$R \cong \begin{bmatrix} \mathbb{M}_{n_1}(D) & \mathbb{M}_{n_1\times n_2}(D) & . & . & . & \mathbb{M}_{n_1\times n_k}(D) \\ 0 & \mathbb{M}_{n_2}(D) & . & . & . & \mathbb{M}_{n_2\times n_k}(D) \\ 0 & 0 & \mathbb{M}_{n_3}(D) & . & . & \mathbb{M}_{n_3\times n_k}(D) \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & \mathbb{M}_{n_k}(D) \end{bmatrix}$$

Conversely, suppose that

$$R \cong \begin{bmatrix} \mathbb{M}_{n_1}(D) & \mathbb{M}_{n_1\times n_2}(D) & . & . & . & \mathbb{M}_{n_1\times n_k}(D) \\ 0 & \mathbb{M}_{n_2}(D) & . & . & . & \mathbb{M}_{n_2\times n_k}(D) \\ 0 & 0 & \mathbb{M}_{n_3}(D) & . & . & \mathbb{M}_{n_3\times n_k}(D) \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & \mathbb{M}_{n_k}(D) \end{bmatrix}$$

where D is a division ring and $n_1, ..., n_k$ are positive integers.

Clearly, R is an indecomposable right non-singular ring. By Eisenbud-Griffith [4], we know that R is an artinian serial ring. Therefore, R is a right Σ-q ring. □

Acknowledgment. We would like to thank the referee for helpful suggestions.

REFERENCES

[1] F. W. Anderson, K. R. Fuller, Rings and Categories of Modules, Second edition, Springer, New York, 1992.

[2] K. I. Beidar, Y. Fong, W.-F. Ke, and S. K. Jain, An Example of a Right q-ring, Israel Journal of Mathematics 127 (2002), 303-316.

[3] K. A. Byrd, Right self-injective rings whose essential ideals are two-sided, Pacific Journal of Mathematics 82 (1979), 23-41.

[4] D. Eisenbud, P. Griffith, The Structure of Serial Rings, Pacific Journal of Mathematics, 36, 1 (1971), 109-121.

[5] A. Facchini, Module Theory: Endomorphism Rings and Direct Sum Decompositions in Some Classes of Modules, Birkhauser, 1998.

[6] C. Faith, Algebra II, Ring Theory, Springer-verlag, Berlin-NewYork, 1976.

[7] K. R. Fuller, On indecomposable injectives over Artinian rings, Pacific J. Math. 22, 1 (1969), 115-135.

[8] K. R. Goodearl, von Neumann Regular Rings, Krieger Publishing Company, Malabar, Florida, 1991.

[9] D. A. Hill, Semi-perfect q-rings, Mathematische Annalen 200 (1973), 113-121.

[10] G. Ivavov, Non-local rings whose ideals are quasi-injective, Bulletin of the Australian Mathematical Society 6 (1972), 45-52.

[11] G. Ivavov, Non-local rings whose ideals are quasi-injective: Addendum, Bulletin of the Australian Mathematical Society 12 (1975), 159-160.

[12] S. K. Jain, S. H. Mohamed, Surjeet Singh, Rings in which each right ideal is quasi-injective, Pacific Journal of Mathematics 31 (1969), 73-79.

[13] R. E. Johnson, E. T. Wong, Quasi-injective modules and irreducible rings, J. London Math. Soc. 36 (1961), 260-268.

[14] A. Koehler, Rings with quasi-injective cyclic modules, Quar. J. Math. Oxford (2), 25 (1974), 51-55.

[15] T. Y. Lam, Lectures on Modules and Rings, GTM, Vol. 189, Springer-Verlag, 1999.

[16] S. H. Mohamed, q-rings with chain conditions, J. London Math. Soc. 2 (1972), 455-460.

[17] T. Nakayama, On Frobeniusean algebras II, Ann. of Math. 42 (1941), 1-21.

[18] G. Puninski, Serial Rings, Kluwer Academic Publishers, 2001.

[19] F. L. Sandomierski, Linearly compact modules and local Morita duality, Ring Theory, Academic Press, New York, 1972.

[20] R. B. Warfield, Jr., Serial rings and finitely presented modules, Journal of Algebra 37 (1975), 187-222.

Department of Mathematics, Ohio University, Athens, Ohio-45701, USA
E-mail address: jain@math.ohiou.edu

Department of Mathematics, Panjab University, Chandigarh, India
E-mail address: ossinghpal@yahoo.co.in

Department of Mathematics and Computer Science, St. Louis University, St. Louis, MO-631003, USA
E-mail address: asrivas3@slu.edu