ESSENTIAL EXTENSIONS OF A DIRECT SUM OF SIMPLE MODULES-II

S. K. JAIN AND ASHISH K. SRIVASTAVA

Dedicated to Robert Wisbauer on his 65th Birthday

Abstract. It is shown that a semi-regular ring R with the property that each essential extension of a direct sum of simple right R-modules is a direct sum of quasi-injective right R-modules is right noetherian.

1. Introduction

The question whether a von-Neumann regular ring R with the property that every essential extension of a direct sum of simple right R-modules is a direct sum of quasi-injective right R-modules is noetherian was considered in [2]. This question was answered in the affirmative under a stronger hypothesis. The purpose of this note is to answer the question in the affirmative even for a more general class of rings, namely, semi-regular, semi-perfect.

2. DEFINITIONS AND NOTATIONS

All rings considered in this paper have unity and all modules are right unital. Let M be an R-module. We denote by $Soc(M)$, the socle of M. We shall write $N \subseteq_e M$ whenever N is an essential submodule of M. A module M is called N-injective, if every R-homomorphism from a submodule L of N to M can be lifted to a R-homomorphism from N to M. A module M is said to be quasi-injective if it is M-injective. A ring R is said to be right $q.f.d.$ if every cyclic right R-module has finite uniform (Goldie) dimension, that is, every direct sum of submodules of a cyclic module has finite number of terms. We shall say that Goldie dimension of N with respect to U, $G\dim_U(N)$, is less than or equal to n, if for any independent family $\{V_j : j \in \mathcal{J}\}$ of nonzero submodules of N such that each V_j is isomorphic to a submodule of U, we have that $|\mathcal{J}| \leq n$. Next, $G\dim_U(N) < \infty$ means that $G\dim_U(N) \leq n$ for some positive integer n. The module N is said to be $q.f.d.$ relative to U if for any factor module $\bar{N}$ of N, $G\dim_U(\bar{N}) < \infty$. A ring R is called von-Neumann regular if every principal right (left) ideal of R is generated by an idempotent. A regular ring is called abelian if all its idempotents are central. A ring R is called semiregular if $R/J(R)$ is von Neumann regular. A ring R is said

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to be a semilocal ring if $R/J(R)$ is semisimple artinian. In a semilocal ring R every set of orthogonal idempotents is finite, and R has only finitely many simple modules up to isomorphism. A ring R is called a q-ring if every right ideal of R is quasi-injective [4]. For any term not defined here, we refer the reader to Wisbauer [6].

3. MAIN RESULTS

Throughout, we will refer to the condition 'every essential extension of a direct sum of simple R-modules is a direct sum of quasi-injective R-modules' as property (*).

We note that the property (*) is preserved under ring homomorphic images.

We first state some of the results that are used throughout the paper.

Lemma 1. *[2] Let R be a ring which satisfies the property (*) and let N be a finitely generated R-module. Then there exists a positive integer n such that for any simple R-module S, we have*

$$G\dim_S(N) \leq n.$$

Lemma 2. *[2] Let R be a right nonsingular ring which satisfies the property (*). Then R has a bounded index of nilpotence.*

The following lemma is a key to the proof of main results.

Lemma 3. *Let R be an abelian regular ring with the property (*). Then R is right noetherian.*

Proof. Recall that an abelian regular ring is duo. Assume R is not right noetherian. Then there exists an infinite family $\{e_iR : e_i = e_i^2, i \in I\}$of independent ideals in R. Now For each $i \in I$, there exists a maximal right ideal M_i such that each $e_iR \nsubseteq M_i$, for otherwise $e_iR \subseteq J(R)$ which is not possible. Set $A = \oplus_{i\in I}e_iR$ and $M = \oplus_{i\in I}e_iM_i$. Note $M \neq R$ and $A/M = (\oplus_{i\in I}e_iR)/(\oplus_{i\in I}e_iM_i)$. So, R/M is a ring with nonzero socle of infinite Goldie dimension. Choose $K/M \subset R/M$ such that $Soc(R/M) \oplus K/M \subset_e R/M$. This implies that $Soc(R/M)$ is essentially embeddable in R/K and so $Soc(R/M) \cong Soc(R/K)$. Obviously, $Soc(R/K) \subset_e R/K$. Set $\bar{R} = R/K$. By (*), $\bar{R} = \bar{e_1}\bar{R} \oplus ... \oplus \bar{e_k}\bar{R}$, where each $\bar{e_i}\bar{R}$ is quasi-injective. Since each $\bar{e_i}$ is a central idempotent, $\bar{e_i}\bar{R}$ is $\bar{e_j}\bar{R}$-injective. Hence, $\bar{e_1}\bar{R} \oplus ... \oplus \bar{e_k}\bar{R}$ is quasi-injective. Thus, $\bar{R} = R/K$ is a right self-injective duo ring and hence R/K is a q-ring. Since R/K is a regular q-ring, $R/K = S \oplus T$, where S is semisimple artinian and T has zero socle (see Theorem 2.18, [4]). But R/K has essential socle. So $T = 0$ and hence R/K is semisimple artinian, which gives a contradiction to the fact that R/K contains an infinite independent family of right ideals. Therefore, R must be right noetherian. This completes the proof. □

Next we claim that if the matrix ring has the property (*) then the base ring also has the property (*). This can be deduced from the following two lemmas whose proofs are standard and given here only for the sake of completeness.

Lemma 4. *If R is a ring with the property (*) and $ReR = R$, then eRe is also a ring with the property (*).*

Proof. We know that if $ReR = R$, then *mod-R* and *mod-eRe* are Morita equivalent under the functors given by $\mathcal{F}$: *mod-R* $\longrightarrow$ *mod-eRe*, $\mathcal{G}$: *mod-eRe* $\longrightarrow$ *mod-R* such that for any M_R, $\mathcal{F}(M) = Me$ and for any module T over eRe, $\mathcal{G}(T) = T \otimes_{eRe} eR$.

Suppose R is a ring with the property (*). Let N be an essential extension of a direct sum of simple eRe-modules $\{S_i : i \in I\}$. This gives, $\oplus_i S_i \otimes_{eRe} eR \subset_e N \otimes_{eRe} eR$. By Morita equivalence each $S_i \otimes_{eRe} eR$ is a simple R-module. Thus, $N \otimes_{eRe} eR$ is an essential extension of a direct sum of simple R-modules. But since R is a ring with the property (*), we have $N \otimes_{eRe} eR = \oplus_i A_i$, where A_i's are quasi-injective R-modules. By Morita equivalence we get that each $A_i e$ is quasi-injective as an eRe-module. Then $N = NeRe = A_1 e \oplus ... \oplus A_n e$ is a direct sum of quasi-injective eRe-modules. Hence eRe is a ring with the property (*). □

As a consequence of the above lemma, we have the following:

Lemma 5. *If $M_n(R)$ is a ring with the property (*), then R is also a ring with the property (*).*

Proof. We have $R \cong e_{11}\mathbb{M}_n(R)e_{11}$ and $\mathbb{M}_n(R)e_{11}\mathbb{M}_n(R) = \mathbb{M}_n(R)$, where e_{11} is a matrix unit. Therefore, the result follows from the Lemma 4. □

Now we are ready to prove the result that answers the question raised in [2].

Theorem 6. *Let R be a regular ring with the property (*). Then R is right noetherian.*

Proof. By Lemma 2, R has bounded index of nilpotence. Hence each primitive factor ring of R is artinian. Therefore, $R \simeq M_n(S)$ for some abelian regular ring S (see Theorem 7.14, [3]). By Lemma 5, S has the property (*). Therefore, by Lemma 3, S is right noetherian. Hence, R is right noetherian. □

We next proceed to generalize the above theorem to semiregular rings. First we prove the following:

Lemma 7. *Let R be a semilocal ring with the property (*). Then R is right noetherian.*

Proof. We claim that R_R is right *q.f.d.* Consider any cyclic module R/I. Suppose there exists an infinite direct sum $A_1/I \oplus A_2/I \oplus ... \subset R/I$, where $\frac{A_i}{I} = \frac{a_iR+I}{I}$. Let M_i/I be a maximal submodule of A_i/I for each i, and set $M/I = \oplus M_i/I$. Then $\frac{A_1}{M_1} \times \frac{A_2}{M_2} \times ... \cong \frac{A_1 \oplus A_2 \oplus ...}{M_1 \oplus M_2 \oplus ...} \subset R/M$. Each A_i/M_i is a simple module. Set $S_i = A_i/M_i$. Since the semilocal ring R/M has only finitely many simple modules upto isomorphism, copies of at least one of the A_i/M_i must appear infinitely many times, and so $G\dim_{S_i}(R/M) = \infty$, for some i. This gives a contradiction to Lemma 1. Therefore, R is right *q.f.d.* Hence, by (Theorem 2.2, [1]), R is right noetherian. □

Corollary 8. *A right self-injective ring with the property (*) is Quasi-Frobenius.*

Theorem 9. *Let R be a semiregular ring with the property (*). Then R is right noetherian.*

Proof. $R/J(R)$ is a von Neumann regular ring with the property (*). Therefore, by Theorem 6, $R/J(R)$ is a right noetherian and hence a semisimple artinian ring. So, R is a semilocal ring. Finally, by Lemma 7, R is right noetherian. □

References

[1] K. I. Beidar and S. K. Jain, When Is Every Module with Essential Socle a Direct Sums of Quasi-Injectives?, Communications in Algebra, 33, 11 (2005), 4251-4258.

[2] K. I. Beidar, S. K. Jain and Ashish K. Srivastava, Essential Extensions of a Direct Sum of Simple Modules, Groups, Rings and Algebras, Contemporary Mathematics, AMS, 420 (2006), 15-23.

[3] K. R. Goodearl, von Neumann Regular Rings, Krieger Publishing Company, Malabar, Florida, 1991.

[4] S. K. Jain, S. H. Mohamed, Surjeet Singh, Rings In Which Every Right Ideal Is Quasi-Injective, Pacific Journal of Mathematics, Vol. 31, No.1 (1969), 73-79.

[5] S. H. Mohamed and B. J. Muller, Continuous and Discrete Modules, Cambridge University Press, 1990.

[6] R. Wisbauer, Foundations of Module and Ring Theory, Gordon and Breach, 1991.

Department of Mathematics, Ohio University, Athens, Ohio-45701, USA
E-mail address: `jain@math.ohiou.edu`

Department of Mathematics, Ohio University, Athens, Ohio-45701, USA
E-mail address: `ashish@math.ohiou.edu`