The Monochromatic Column Problem

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Abstract

Let p_1, p_2,..., p_n be a sequence of n pairwise coprime positive integers, and let $P = p_1 p_2 ... p_n$. Let $0, 1, ..., m-1$ be a sequence of m different colors. Let A be an $n \times mP$ matrix of colors in which row i consists of blocks of p_i consecutive entries of the same color, with colors 0 through $m-1$ repeated cyclically. The Monochromatic Column problem is to determine the number of columns of A in which every entry is the same color. The solution for $m = 3$ colors is presented and proved.

Keywords: Monochromatic Column; Chinese Remainder Theorem
2000 Mathematics Subject Classification. 05A15, 11A07

1 Introduction

Motivated by a question raised in [1] regarding finding an alignment with optimal score according to a given scoring scheme for given n sequences of characters from a fixed alphabet, Nagpaul and Jain in [2], introduce the following Monochromatic Column problem. Let p_1, p_2,..., p_n be a sequence of n pairwise coprime positive integers, and let $P = p_1 p_2 ... p_n$. Let $0, 1, ..., m-1$ be a sequence of m colors. Let A be an $n \times mP$ matrix of colors in which row i consists of blocks of p_i consecutive entries of the same color, with colors 0 through $m-1$ repeated cyclically (see Section 2 for the precise definition). The Monochromatic Column problem is to count the number of columns of A in which every entry in the column is the same color.

In [2], Nagpaul and Jain solve the Monochromatic Column problem for the case of $m = 2$ colors. In the case of $m > 2$ colors, they solve the problem under the constraints that all rows begin with the same color and that, for $1 \leq i_1 < i_2 \leq n$, the integers p_{i_1} and p_{i_2} are congruent to each other modulo

m. Here the Monochromatic Column problem for $m = 3$ colors, which are necessarily 0, 1, and 2, is solved.

To illustrate, let $n = 2$, $p_1 = 2$, and $p_2 = 3$. Then $P = 6$ and any matrix will be 2×18. One such matrix is the following:

$$A = \begin{pmatrix} 0 & 0 & 1 & 1 & 2 & 2 & 0 & 0 & 1 & 1 & 2 & 2 & 0 & 0 & 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 2 & 2 & 2 & 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 & 0 & 0 & 0 \end{pmatrix}.$$

For this A, the number of monochromatic columns is 6. Note that the first row starts with a block of color 0, while the second row starts with a block of color 1. Since there are 3 choices of starting color for each row, there are $3^2 = 9$ matrices possible.

Here, the Monochromatic Column problem for $m = 3$ without constraints is solved (Section 3). Section 4 counts the monochromatic column blocks, where consecutive monochromatic columns of the same color form monochromatic blocks. In Section 5, we discuss the monochromatic column problem for arbitrary number of colors under certain special cases. Section 2 lays out the terminology and notation.

2 Terminology and Notation

Let m be a positive integer. The colors for m are the integers $0, 1, ..., m-1$. An $r \times s$ color matrix is an $r \times s$ matrix $A = (a_{ij})$ in which every entry is one of the m colors. Column j of A is a *monochromatic column* if $a_{i_1 j} = a_{i_2 j}$, for all i_1 and i_2 satisfying $1 \leq i_1 < i_2 \leq r$. For a positive integer q, row i of A is q-blocked with initial color ρ if $q|s$ and, for $1 \leq j \leq s$, the i, j entry is the color

$$a_{ij} = \left(\left\lfloor \frac{j-1}{q} \right\rfloor + \rho \right) \bmod m.$$

A is the $(q_1, q_2, \ldots, q_n; \rho_1, \rho_2, \ldots, \rho_n)$ *color matrix of width* s if, for every i satisfying $1 \leq i \leq n$, we have $q_i|s$ and row i of A is q_i-blocked with initial color ρ_i. The sequence $p_1, p_2, \ldots, p_n$ of positive integers is coprime if p_{i_1} and p_{i_2} are coprime, for all i_1 and i_2 satisfying $1 \leq i_1 < i_2 \leq n$. Let $P = p_1 p_2 \cdots p_n$. Note that the permutation of rows will not affect the number of monochromatic columns. The notation $(m; p_1, p_2, \ldots, p_n; c_1, c_2, \ldots, c_r; t_1, ..., t_{r-1})$ will mean that rows $t_{k-1} + 1$ to t_k start with color c_k for $1 \leq k \leq r$ defining $t_0 = 0$ and $t_r = n$. The Monochromatic Column problem is to determine the count $N(m; p_1, p_2, \ldots, p_n; c_1, c_2, \ldots, c_r; t_1, ..., t_{r-1})$ of monochromatic columns in the $(m; p_1, p_2, \ldots, p_n; c_1, c_2, \ldots, c_r; t_1, ..., t_{r-1})$ color matrix A of width mP. Given a color matrix A of width mP, from here out just referred to as a canonical color matrix, $N(A)$ will be the number of monochromatic columns in A.

For $i = 1, 2, ..., n$ and $j = 1, 2, ..., mP$, define

$$b_{ij} = \left\lfloor \frac{j-1}{p_i} \right\rfloor$$

and

$$k_{ij} = j - b_{ij} p_i.$$

Then, $1 \leq k_{ij} \leq p_i$, and the color in entry i, j is

$$a_{ij} = (b_{ij} + \rho_i) \bmod m$$

where ρ_i is the initial color of row i.

Define a column j to be an *h-bichromatic column* if for fixed h, $a_{ij} = c_u$ for $1 \leq i \leq h$ and $a_{ij} = c_l$ for $h+1 \leq i \leq n$, for fixed $c_u, c_l \in \{0, 1, ..., m-1\}$. In the case $c_u = c_l$, the column j becomes a monochromatic column. Define a set of consecutive monochromatic columns $\{j, \ldots, j+t\}$ for some t to be a *monochromatic block* if $a_{1j} = a_{1k}$ for all k satisfying $j \leq k \leq j+t$ and if $j \neq 1$, column $j-1$ is not monochromatic and if $j+t \neq n$, column $j+t+1$ is not monochromatic.

3 Counting Monochromatic Columns in 3 Colors

To solve the Monochromatic Column problem for $m = 3$, we consider three cases. The first case requires that the p_i are pairwise congruent modulo 3.

Lemma 1 *Assume $p_1, ..., p_n$ are coprime, $p_i \equiv s (\bmod 3)$ for $1 \leq i \leq n$ and fixed $s \in \{1, 2\}$ and t_0, t_1, t_2, t_3 satisfy $0 = t_0 \leq t_1 \leq t_2 \leq t_3$. Then*

$$N(3; p_1, p_2, \ldots, p_n; 0, 1, 2; t_1, t_2) = 3 \sum_{\beta=0}^{2} \prod_{\rho=0}^{2} \prod_{i=t_\rho+1}^{t_{\rho+1}} \left(\frac{p_i - s}{3} + G_{\beta,\rho,s} \right)$$

where $G_{\beta,\rho,s} = \left\lfloor \frac{s - ((\beta+\rho s) \bmod 3)}{3} \right\rfloor + \left\lfloor \frac{((\beta+\rho s) \bmod 3)+2}{3} \right\rfloor$.

Proof. Let $P = p_1 p_2 \cdots p_n$. Let A be the $(3; p_1, p_2, ..., p_n; 0, 1, 2; t_1, t_2)$ canonical color matrix. Denote the initial color of row i by ρ_i. By definition,

$$a_{ij} = \left(\left\lfloor \frac{j-1}{p_i} \right\rfloor + \rho_i \right) \bmod 3.$$

Let i_1 and i_2 satisfy $1 \leq i_1 \leq i_2 \leq n$. Observe that, for all $j \in \{1, 2, ..., P\}$, we have

$$\begin{aligned}
(a_{i_1,j} - a_{i_1,P+j}) \bmod 3 &= \left(\left\lfloor \tfrac{j-1}{p_{i_1}} \right\rfloor + \rho_{i_1} - \left\lfloor \tfrac{P+j-1}{p_{i_1}} \right\rfloor - \rho_{i_1} \right) \bmod 3 \\
&= \left(\left\lfloor \tfrac{j-1}{p_{i_1}} \right\rfloor - \tfrac{P}{p_{i_1}} - \left\lfloor \tfrac{j-1}{p_{i_1}} \right\rfloor \right) \bmod 3 \\
&= \left(\left\lfloor \tfrac{j-1}{p_{i_2}} \right\rfloor - \tfrac{P}{p_{i_2}} - \left\lfloor \tfrac{j-1}{p_{i_2}} \right\rfloor \right) \bmod 3 \\
&= \left(\left\lfloor \tfrac{j-1}{p_{i_2}} \right\rfloor + \rho_{i_2} - \left\lfloor \tfrac{P+j-1}{p_{i_2}} \right\rfloor - \rho_{i_2} \right) \bmod 3 \\
&= (a_{i_2,j} - a_{i_2,P+j}) \bmod 3
\end{aligned}$$

since $\frac{P}{p_i} \equiv \frac{P}{p_j} (\bmod 3)$. Similarly,

$$(a_{i_1,j} - a_{i_1,2P+j}) \bmod 3 = (a_{i_2,j} - a_{i_2,2P+j}) \bmod 3.$$

Fix any integer j, where $1 \leq j \leq P$. Then column j being monochromatic is equivalent to column $P+j$ being monochromatic, which is equivalent to column $2P+j$ being monochromatic. Hence, it suffices to count the number of monochromatic columns in the first P columns of A and multiply by 3. Now, let $1 \leq i \leq n$ and $1 \leq j \leq P$. Since $k_{ij} = j - b_{ij}p_i = j - \left\lfloor \frac{j-1}{p_i} \right\rfloor p_i$, the color in entry i, j is

$$\begin{aligned} a_{ij} &= \left(\left\lfloor \frac{j-1}{p_i} \right\rfloor + \rho_i \right) \bmod 3 \\ &= \left(\frac{j-k_{ij}}{p_i} + \rho_i \right) \bmod 3 \end{aligned}$$

Conversely, let $(v_1, v_2, ..., v_n)$ be any n-tuple such that $1 \leq v_i \leq p_i$. Since $p_1, p_2, ..., p_n$ are coprimes, the Chinese Remainder Theorem guarantees a unique $j \in \{1, 2, \ldots, P\}$ such that $j \equiv v_i (\bmod\, p_i)$ for each $i \in \{1, 2, \ldots, n\}$. Thus, the mapping of $j \in \{1, 2, ..., P\}$ to $(k_{1j}, k_{2j}, ..., k_{nj})$ is a 1-1 correspondence.

Let $j \in \{1, 2, ..., P\}$ be such that column j of A is monochromatic with common color $a_{1j} = a_{2j} = \cdots = a_{nj}$. Then, for $i \in \{1, 2, ..., n\}$, we have

$$\begin{aligned} a_{1j} &= \left(\frac{j-k_{ij}}{p_i} + \rho_i \right) \bmod 3 \\ k_{ij} &\equiv j - a_{1j}p_i + \rho_i p_i (\bmod\, 3) \\ &\equiv j - a_{1j}s + \rho_i s (\bmod\, 3) \end{aligned}$$

since $p_i \equiv s (\bmod\, 3)$. Define $\beta_j = j - a_{1j}s$. Then, column j is monochromatic if and only if

$$k_{ij} \equiv \beta_j + \rho_i s (\bmod\, 3),$$

for all $i \in \{1, 2, ..., n\}$. Let $\gamma_{ij} = (\beta_j + \rho_i s) \bmod 3$. Column j is monochromatic if and only if $k_{ij} \equiv \gamma_{ij} (\bmod\, 3)$ meaning that setting $\alpha_{ij} = \frac{(k_{ij}-\gamma_{ij})}{3}$, then $1 \leq 3\alpha_{ij} + \gamma_{ij} \leq p_i$ or, equivalently, $\left\lceil \frac{1-\gamma_{ij}}{3} \right\rceil \leq \alpha_{ij} \leq \left\lfloor \frac{p_i-\gamma_{ij}}{3} \right\rfloor = \frac{p_i-s}{3} - \left\lceil \frac{\gamma_{ij}-s}{3} \right\rceil$. For a fixed γ_{ij} there are

$$\begin{aligned} \frac{p_i-s}{3} + \left\lfloor \frac{s-\gamma_{ij}}{3} \right\rfloor + \left\lfloor \frac{\gamma_{ij}-1}{3} \right\rfloor + 1 &= \frac{p_i-s}{3} + \left\lfloor \frac{s-\gamma_{ij}}{3} \right\rfloor + \left\lfloor \frac{\gamma_{ij}+2}{3} \right\rfloor \\ &= \frac{p_i-s}{3} + \left\lfloor \frac{s-((\beta_j+\rho_i s) \bmod 3)}{3} \right\rfloor + \left\lfloor \frac{((\beta_j+\rho_i s) \bmod 3)+2}{3} \right\rfloor \end{aligned}$$

solutions for α_{ij} and hence for k_{ij}. Summing over the three possibilities for β_j, multiplying the numbers of solutions for each row, and multiplying the sum by 3, we obtain the desired expression for the count of monochromatic columns in A.

$$
\begin{aligned}
N(A) &= 3\sum_{\beta=0}^{2}\left(\prod_{i=1}^{t_1}\frac{p_i-s}{3}+\left\lfloor\frac{s-(\beta \bmod 3)}{3}\right\rfloor+\left\lfloor\frac{(\beta \bmod 3)+2}{3}\right\rfloor\right)\\
&\quad\left(\prod_{i=t_1+1}^{t_2}\frac{p_i-s}{3}+\left\lfloor\frac{s-((\beta+s) \bmod 3)}{3}\right\rfloor+\left\lfloor\frac{((\beta+s) \bmod 3)+2}{3}\right\rfloor\right)\\
&\quad\left(\prod_{i=t_2+1}^{n}\frac{p_i-s}{3}+\left\lfloor\frac{s-((\beta+2s) \bmod 3)}{3}\right\rfloor+\left\lfloor\frac{((\beta+2s) \bmod 3)+2}{3}\right\rfloor\right)\\
&= 3\sum_{\beta=0}^{2}\prod_{\rho=0}^{2}\prod_{i=t_\rho+1}^{t_{\rho+1}}\frac{p_i-s}{3}+\left\lfloor\frac{s-((\beta+\rho s) \bmod 3)}{3}\right\rfloor+\left\lfloor\frac{((\beta+\rho s) \bmod 3)+2}{3}\right\rfloor.
\end{aligned}
$$

■

Next, we consider the case where for all $i \in \{1,\ldots,n\}$ either $p_i \equiv 1 (\bmod 3)$ or $p_i \equiv 2 (\bmod 3)$.

Lemma 2 *For fixed h and n such that $1 \le h < n$, assume $p_1, ..., p_n$ are coprimes, $p_i \equiv 1(\bmod 3)$ for $1 \le i \le h$, $p_i \equiv 2(\bmod 3)$ for $h+1 \le i \le n$ and t_0, t_1, t_2, t_3, t_4, t_5, t_6 satisfy $0 = t_0 \le t_1 \le t_2 \le t_3 = h \le t_4 \le t_5 \le t_6$. Then*

$$
\begin{aligned}
&N(3; p_1, p_2, \ldots, p_n; 0,1,2,0,1,2; t_1,t_2,t_3,t_4,t_5) \quad = \\
&\sum_{\beta_1=0}^{2}\sum_{\beta_2=0}^{2}\prod_{s=1}^{2}\prod_{\rho=0}^{2}\prod_{i=t_{3(s-1)+\rho}+1}^{i=t_{3(s-1)+\rho+1}}\left(\frac{p_i-s}{3}+G_{\beta_s,\rho,s}\right)
\end{aligned}
$$

where $G_{\beta,\rho,s} = \left\lfloor\frac{s-((\beta+\rho s) \bmod 3)}{3}\right\rfloor + \left\lfloor\frac{((\beta+\rho s) \bmod 3)+2}{3}\right\rfloor$.

Proof. Let $P = p_1, ..., p_n$. Let A be the $(3; p_1, p_2, \ldots, p_n; 0,1,2,0,1,2; t_1,t_2,t_3,t_4,t_5)$ canonical color matrix. Denote the initial color of row i by ρ_i. By definition, $b_{ij} = \lfloor\frac{j-1}{p_i}\rfloor$, $a_{ij} = (\lfloor\frac{j-1}{p_i}\rfloor + \rho_i) \bmod 3$ and $k_{ij} = j - b_{ij}p_i$. Let i_1 and i_2 satisfy $1 \le i_1 \le i_2 \le h$. Observe that, for all $j \in \{1,2,...,P\}$, we have

$$
\begin{aligned}
(a_{i_1,j}-a_{i_1,P+j}) \bmod 3 &= \left(\left\lfloor\tfrac{j-1}{p_{i_1}}\right\rfloor+\rho_{i_1}-\left\lfloor\tfrac{P+j-1}{p_{i_1}}\right\rfloor-\rho_{i_1}\right) \bmod 3\\
&= \left(\left\lfloor\tfrac{j-1}{p_{i_1}}\right\rfloor-\tfrac{P}{p_{i_1}}-\left\lfloor\tfrac{j-1}{p_{i_1}}\right\rfloor\right) \bmod 3\\
&= \left(\left\lfloor\tfrac{j-1}{p_{i_2}}\right\rfloor-\tfrac{P}{p_{i_2}}-\left\lfloor\tfrac{j-1}{p_{i_2}}\right\rfloor\right) \bmod 3\\
&= \left(\left\lfloor\tfrac{j-1}{p_{i_2}}\right\rfloor+\rho_{i_2}-\left\lfloor\tfrac{P+j-1}{p_{i_2}}\right\rfloor-\rho_{i_2}\right) \bmod 3\\
&= (a_{i_2,j}-a_{i_2,P+j}) \bmod 3
\end{aligned}
$$

since $\frac{P}{p_{i_1}} \equiv \frac{P}{p_{i_2}}(\bmod 3)$. Similarly,

$$
(a_{i_1,j}-a_{i_1,2P+j}) \bmod 3 = (a_{i_2,j}-a_{i_2,2P+j}) \bmod 3.
$$

We can show the same result for i_1 and i_2 that satisfy $h+1 \leq i_1 \leq i_2 \leq n$. Fix any integer j, where $1 \leq j \leq P$. Then column j being h-bichromatic is equivalent to column $P+j$ being h-bichromatic, which is equivalent to column $2P+j$ being h-bichromatic.

Next, we show that if j is h-bihromatic then one and only one of the columns j, $j+P$ and $j+2P$ is a monochromatic column. Let $j \in \{1, \ldots, P\}$ and assume column j is h-bichromatic. Let us denote by ordered pair (c_u, c_l) the entries of a h-bichromatic column where $a_{ij} = c_u$ for all $i \in \{1, \ldots, h\}$ and $a_{ij} = c_l$ for all $i \in \{h+1, \ldots, n\}$. Note, that if $c_u = c_l$ then we have a monochromatic column. Let $r_i = \frac{P}{p_i} \bmod 3$. Then $r_i = 2^{n-h} \bmod 3$ when $i = 1, ..., h$, and $r_i = 2^{n-h-1} \bmod 3$ when $i = h+1, ..., n$. If $(n-h)$ is even, the entries of h-bichromatic columns $j+P$ and $j+2P$ are $((c_u+1) \bmod 3, (c_l+2) \bmod 3)$ and $((c_u+2) \bmod 3, (c_l+1) \bmod 3)$ respectively and if $(n-h)$ is odd, the entries of h-bichromatic columns $j+P$ and $j+2P$ are $((c_u+2) \bmod 3, (c_l+1) \bmod 3)$ and $((c_u+1) \bmod 3, (c_l+2) \bmod 3)$ respectively. In either case, for one and only one of these pairs of the form (a, b), $a = b$, so one and only one of the h-bichromatic columns j, $j+P$ and $j+2P$ is a monochromatic column. Hence, the number of h-bichromatic columns in the first P columns is equal to the number of monochromatic columns in A. So, it suffices to count the number of h-bichromatic columns in the first P columns.

Let $j \in \{1, 2, ..., P\}$ such that the column j of A is h-bichromatic with $a_{1j} = a_{2j} = ... = a_{hj}$ and $a_{h+1,j} = a_{h+2,j} = ... = a_{nj}$. Then, for $1 \leq i \leq h$, we have

$$
\begin{aligned}
a_{1j} &= \left(\tfrac{j-k_{ij}}{p_i} + \rho_i\right) \bmod 3 \\
k_{ij} &\equiv j - a_{1j}p_i + \rho_i p_i \pmod 3 \\
&\equiv j - a_{1j} + \rho_i \pmod 3
\end{aligned}
$$

since $p_i \equiv 1 \pmod 3$ and for $h+1 \leq i \leq n$, we have

$$
\begin{aligned}
a_{h+1,j} &= \left(\tfrac{j-k_{ij}}{p_i} + \rho_i\right) \bmod 3 \\
k_{ij} &\equiv j - a_{h+1,j}p_i + \rho_i p_i \pmod 3 \\
&\equiv j - 2a_{h+1,j} + 2\rho_i \pmod 3
\end{aligned}
$$

since $p_i \equiv 2 \pmod 3$.

Define $\beta_j = j - a_{1j}$ and $\beta_j^{'} = j - 2a_{h+1,j}$. Then, column j is h-bichromatic if and only if $k_{ij} \equiv (\beta_j + \rho_i) \pmod 3$ for all $i \in \{1, 2, ..., h\}$ and $k_{ij} \equiv (\beta_j^{'} + 2\rho_i) \pmod 3$ for all $i \in \{h+1, ..., n\}$.

Let $\gamma_{ij} = (\beta_j + \rho_i) \pmod 3$ for $i \in \{1, 2, ..., h\}$ and $\gamma_{ij} = (\beta_j^{'} + 2\rho_i) \pmod 3$ for $i \in \{h+1, \ldots, n\}$. Column j is h-bichromatic if and only if $k_{ij} \equiv \gamma_{ij} (\bmod 3)$ meaning setting $\alpha_{ij} = \frac{(k_{ij}-\gamma_{ij})}{3}$ then $1 \leq 3\alpha_{ij} + \gamma_{ij} \leq p_i$ or equivalently, for $i \in \{1, 2, ..., h\}$, $\left\lceil \frac{1-\gamma_{ij}}{3} \right\rceil \leq \alpha_{ij} \leq \left\lfloor \frac{p_i - \gamma_{ij}}{3} \right\rfloor = \frac{p_i - 1}{3} - \left\lceil \frac{\gamma_{ij} - 1}{3} \right\rceil$ and for $i \in \{h+1, ..., n\}$,

$\left\lceil \frac{1-\gamma_{ij}}{3} \right\rceil \le \alpha_{ij} \le \left\lfloor \frac{p_i-\gamma_{ij}}{3} \right\rfloor = \frac{p_i-2}{3} - \left\lceil \frac{\gamma_{ij}-2}{3} \right\rceil$. For a fixed γ_{ij} there are

$$\begin{aligned} \frac{p_i-1}{3} + \left\lfloor \frac{1-\gamma_{ij}}{3} \right\rfloor + \left\lfloor \frac{\gamma_{ij}-1}{3} \right\rfloor + 1 &= \frac{p_i-1}{3} + \left\lfloor \frac{1-\gamma_{ij}}{3} \right\rfloor + \left\lfloor \frac{\gamma_{ij}+2}{3} \right\rfloor \\ &= \frac{p_i-1}{3} + \left\lfloor \frac{1-((\beta_j+\rho_i) \bmod 3)}{3} \right\rfloor + \left\lfloor \frac{((\beta_j+\rho_i) \bmod 3)+2}{3} \right\rfloor \end{aligned}$$

solutions for α_{ij} and hence for k_{ij} if $i \in \{1, ..., h\}$ and

$$\begin{aligned} \frac{p_i-s}{3} + \left\lfloor \frac{2-\gamma_{ij}}{3} \right\rfloor + \left\lfloor \frac{\gamma_{ij}-1}{3} \right\rfloor + 1 &= \frac{p_i-2}{3} + \left\lfloor \frac{2-\gamma_{ij}}{3} \right\rfloor + \left\lfloor \frac{\gamma_{ij}+2}{3} \right\rfloor \\ &= \frac{p_i-2}{3} + \left\lfloor \frac{2-((\beta_j^{'}+2\rho_i) \bmod 3)}{3} \right\rfloor + \left\lfloor \frac{((\beta_j^{'}+2\rho_i) \bmod 3)+2}{3} \right\rfloor \end{aligned}$$

solutions for α_{ij} and hence for k_{ij} if $i \in \{h+1, ..., n\}$. By similar arguments as in proof of Lemma 1, we know that there is a 1-1 correspondence between $j \in \{1, 2, ..., P\}$ and $(k_{1j}, k_{2j}, ..., k_{nj})$. Summing over the three possibilities for both β_j and $\beta_j^{'}$, multiplying the number of solutions for each row, we obtain the count of h-bichromatic columns in the first P columns of A and hence the desired expression for the count of monochromatic columns in A.

$$\begin{aligned} N(A) \quad = \quad & \sum_{\beta=0}^{2} \sum_{\beta'=0}^{2} \left(\prod_{i=1}^{t_1} \frac{p_i-1}{3} + \left\lfloor \frac{1-(\beta \bmod 3)}{3} \right\rfloor + \left\lfloor \frac{(\beta \bmod 3)+2}{3} \right\rfloor \right) \\ & \left(\prod_{i=t_1+1}^{t_2} \frac{p_i-1}{3} + \left\lfloor \frac{1-((\beta+1) \bmod 3)}{3} \right\rfloor + \left\lfloor \frac{((\beta+1) \bmod 3)+2}{3} \right\rfloor \right) \\ & \left(\prod_{i=t_2+1}^{t_3} \frac{p_i-1}{3} + \left\lfloor \frac{1-((\beta+2) \bmod 3)}{3} \right\rfloor + \left\lfloor \frac{((\beta+2) \bmod 3)+2}{3} \right\rfloor \right) \\ & \left(\prod_{i=t_3+1}^{t_4} \frac{p_i-2}{3} + \left\lfloor \frac{2-(\beta^{'} \bmod 3)}{3} \right\rfloor + \left\lfloor \frac{(\beta^{'} \bmod 3)+2}{3} \right\rfloor \right) \\ & \left(\prod_{i=t_4+1}^{t_5} \frac{p_i-2}{3} + \left\lfloor \frac{2-((\beta^{'}+1) \bmod 3)}{3} \right\rfloor + \left\lfloor \frac{((\beta^{'}+1) \bmod 3)+2}{3} \right\rfloor \right) \\ & \left(\prod_{i=t_5+1}^{n} \frac{p_i-2}{3} + \left\lfloor \frac{2-((\beta^{'}+2) \bmod 3)}{3} \right\rfloor + \left\lfloor \frac{((\beta^{'}+2) \bmod 3)+2}{3} \right\rfloor \right). \end{aligned}$$

Rewriting β as β_1 and $\beta^{'}$ as β_2 we get

$$N(A) \quad = \quad \sum_{\beta_1=0}^{2} \sum_{\beta_2=0}^{2} \prod_{s=1}^{2} \prod_{\rho=0}^{2} \prod_{i=t_{3(s-1)+\rho}+1}^{t_{3(s-1)+\rho+1}} \frac{p_i-s}{3} + \left\lfloor \frac{s-((\beta_s+\rho s) \bmod 3)}{3} \right\rfloor + \left\lfloor \frac{((\beta_s+\rho s) \bmod 3)+2}{3} \right\rfloor .$$

■

The final case is where for some k, $3 \mid p_k$. Of course under the assumption that the p_i are coprime there is at most one such k.

Lemma 3 *For $n \geq 2$ assume $p_1, \ldots, p_n$ are coprime, $3 \mid p_n$, $\rho_n \in \{0, 1, 2\}$, and t_0, t_1, t_2, t_3, t_4, t_5, t_6 satisfy $0 = t_0 \leq t_1 \leq t_2 \leq t_3 \leq t_4 \leq t_5 \leq t_6 - 1$. Then*

$$N(3; p_1, ..., p_n; 0, 1, 2, 0, 1, 2, \rho_n; t_1, t_2, t_3, t_4, t_5, t_6) =$$
$$\frac{p_n}{3} N(3; p_1, ..., p_{n-1}; 0, 1, 2, 0, 1, 2; t_1, t_2, t_3, t_4, t_5).$$

Proof. Let $P = p_1 p_2 \cdots p_n$. Let A be the $(3; p_1, p_2, \ldots, p_n; 0, 1, 2, 0, 1, 2, \rho_n; t_1, t_2, t_3, t_4, t_5, t_6)$ canonical color matrix. Denote the initial color of row i by ρ_i. By definition, $b_{ij} = \lfloor \frac{j-1}{p_i} \rfloor$, $a_{ij} = (\lfloor \frac{j-1}{p_i} \rfloor + \rho_i) \bmod 3$ and $k_{ij} = j - b_{ij} p_i$

Observe that, for all $i \in \{1, 2, ..., n-1\}$ and $j \in \{1, 2, ..., P\}$, we have

$$\begin{aligned}
(a_{i,j} - a_{i,P+j}) \bmod 3 &= \left(\left\lfloor \frac{j-1}{p_i} \right\rfloor + \rho_i - \left\lfloor \frac{P+j-1}{p_i} \right\rfloor - \rho_i \right) \bmod 3 \\
&= \left(\left\lfloor \frac{j-1}{p_i} \right\rfloor - \frac{P}{p_i} - \left\lfloor \frac{j-1}{p_i} \right\rfloor \right) \bmod 3 \\
&= \left(-\frac{P}{p_i} \right) \bmod 3 \\
&= 0
\end{aligned}$$

since $3 \mid \frac{P}{p_i}$. Similarly,

$$(a_{i,j} - a_{i,2P+j}) \bmod 3 = 0.$$

Also,

$$\begin{aligned}
(a_{n,j} - a_{n,P+j}) \bmod 3 &= \left(\left\lfloor \frac{j-1}{p_n} \right\rfloor + \rho_n - \left\lfloor \frac{P+j-1}{p_n} \right\rfloor - \rho_n \right) \bmod 3 \\
&= \left(\left\lfloor \frac{j-1}{p_n} \right\rfloor - \frac{P}{p_n} - \left\lfloor \frac{j-1}{p_n} \right\rfloor \right) \bmod 3 \\
&= \left(-\frac{P}{p_n} \right) \bmod 3 \in \{1, 2\}
\end{aligned}$$

since $3 \nmid \frac{P}{p_n}$. Let $r = \left(-\frac{P}{p_n} \right) \bmod 3$. Similarly,

$$(a_{n,j} - a_{n,2P+j}) \bmod 3 = (2r) \bmod 3 \in \{1, 2\}.$$

Fix any integer j, where $1 \leq j \leq P$. Then column j being $(n-1)$-bichromatic is equivalent to column $P + j$ being $(n-1)$-bichromatic, which is equivalent to column $2P + j$ being $(n-1)$-bichromatic. It is easy to see also that one and only one of columns j, $P + j$ or $2P + j$ is monochromatic. So, the number of monochromatic columns in A is equal to the number of $(n-1)$-bichromatic columns in the first P columns of A.

Let B be the $(3; p_1, p_2, \ldots, p_{n-1}; 0, 1, 2, 0, 1, 2; t_1, t_2, t_3, t_4, t_5)$ canonical color matrix. Then the first $n-1$ rows of A can be viewed as p_n copies of matrix B side by side. So the number of $(n-1)$-bichromatic columns in A is $p_n N(B)$ and hence $N(A) = \frac{p_n}{3} N(B)$. ■

Using these 3 lemmas then, the general case can be solved.

Theorem 4 *Let $p_1, ..., p_n$ be coprime integers, $\rho_1, \ldots, \rho_n \in \{0, 1, 2\}$ and let $P = p_1 p_2 ... p_n$. Let A be the color matrix $(p_1, \ldots, p_n; \rho_1, \ldots, \rho_n)$ of width $3P$. Then the number of Monochromatic Columns can be determined. The only cases that exist are*

(1) $p_i \equiv s (\bmod 3)$ *for* $i = 1, ..., n$ *and for fixed* $s \in \{0, 1, 2\}$.

(2) $p_i \equiv 1(\bmod 3)$ *for* $i = 1, ..., h$ *and* $p_i \equiv 2(\bmod 3)$ *for* $i = h+1, ..., n$ *where* $n \geq 2$ *and* $1 \leq h < n$.

(3) $p_n \equiv 0(\bmod 3)$ *and* $p_i \equiv s(\bmod 3)$ *for* $1 \leq i < n$ *and for fixed* $s \in \{1, 2\}$ *where* $n \geq 2$.

(4) $p_n \equiv 0(\bmod 3)$, $p_i \equiv 1(\bmod 3)$ *for* $i = 1, ..., h$ *and* $p_i \equiv 2(\bmod 3)$ *for* $i = h+1, ..., n-1$ *where* $n \geq 3$ *and* $1 \leq h < n-1$.

4 Monochromatic blocks

In this section, the number of monochromatic blocks is computed in the setting of Lemma 1.

Define $B(m; p_1, p_2, \ldots, p_n; c_1, c_2, \ldots, c_r; t_1, ..., t_{r-1})$ to be the number of monochromatic blocks in the canonical color matrix $(m; p_1, p_2, \ldots, p_n; c_1, c_2, \ldots, c_r; t_1, ..., t_{r-1})$ of width mP.

Define $B_q(m; p_1, p_2, \ldots, p_n; c_1, c_2, \ldots, c_r; t_1, ..., t_{r-1})$ to be the number of monochromatic blocks of width q in the canonical color matrix $(m; p_1, p_2, \ldots, p_n; c_1, c_2, \ldots, c_r; t_1, ..., t_{r-1})$ of width mP.

Define $N'(m; p_1, p_2, \ldots, p_n; c_1, c_2, \ldots, c_r; t_1, ..., t_{r-1})$ to be the number of monochromatic columns such that for a monochromatic column j, $k_{ij} \neq 1$ for all $1 \leq i \leq n$ in the canonical color matrix $(m; p_1, p_2, \ldots, p_n; c_1, c_2, \ldots, c_r; t_1, ..., t_{r-1})$.

Lemma 5 *Assume $p_1, ..., p_n$ are pairwise coprime, $p_i \equiv s(\bmod 3)$ for $1 \leq i \leq n$ and fixed $s \in \{1, 2\}$, $p = min\{p_1, \ldots, p_n\}$ and t_0, t_1, t_2, t_3 satisfy $0 = t_0 \leq t_1 \leq t_2 \leq t_3$. Then*

1.

$$\begin{aligned} B(3; p_1, p_2, ..., p_n; 0, 1, 2; t_1, t_2) &= 3 \sum_{\beta=0}^{2} \prod_{\rho=0}^{2} \prod_{i=t_\rho+1}^{t_{\rho+1}} \left(\frac{p_i - s}{3} + G_{\beta,\rho,s} \right) \\ &- 3 \sum_{\beta=0}^{2} \prod_{\rho=0}^{2} \prod_{i=t_\rho+1}^{t_{\rho+1}} \left(\frac{p_i - s}{3} + H_{\beta,\rho,s} \right) \end{aligned}$$

where

$$G_{\beta,\rho,s} = \left\lfloor \frac{s - ((\beta + \rho s) \bmod 3)}{3} \right\rfloor + \left\lfloor \frac{((\beta + \rho s) \bmod 3) + 2}{3} \right\rfloor$$

and

$$H_{\beta,\rho,s} = \left\lfloor \frac{s-((\beta+\rho s) \bmod 3)}{3} \right\rfloor + \left\lfloor \frac{((\beta+\rho s) \bmod 3)+1}{3} \right\rfloor .$$

2.

$$B_p(3;p_1,p_2,\ldots,p_n;0,1,2;t_1,t_2) = \begin{cases} 3\prod_{i=1}^{t_1}(q_i+1)\prod_{i=t_1+1}^{t_2}(q_i)\prod_{i=t_2+1}^{n}(q_i) & \text{for } p \in \{p_1,...,p_{t_1}\} \\ 3\prod_{i=1}^{t_1}(q_i)\prod_{i=t_1+1}^{t_2}(q_i+1)\prod_{i=t_2+1}^{n}(q_i) & \text{for } p \in \{p_{t_1},...,p_{t_2}\} \\ 3\prod_{i=1}^{t_1}(q_i)\prod_{i=t_1+1}^{t_2}(q_i)\prod_{i=t_2+1}^{n}(q_i+1) & \text{for } p \in \{p_{t_2},...,p_n\} \end{cases}$$

where $q_i = \frac{p_i-p}{3}$.

Proof. As usual, let $P = p_1p_2\cdots p_n$. Let A be the $(3;p_1,p_2,...,p_n;0,1,2;t_1,t_2)$ canonical color matrix. Denote the initial color of row i by ρ_i. As in Lemma 1, only the number of monochromatic blocks in the first P columns of matrix A need to be computed to know the total in matrix A. This is due to the symmetry noted in Lemma 1 and to the fact that for any canonical color matrix, $a_{1,P} \neq a_{1,P+1}$ meaning columns P and $P+1$ cannot be in the same monochromatic block. Similarly, this can be seen for columns $2P$ and $2P+1$. Let j be the first column of a monochromatic block. If $j \neq 1$, column $a_{i,j} \neq a_{i,j-1}$ for some i. In either case, $k_{ij} = 1$ for some i. The number of monochromatic blocks then is equal to the number of monochromatic columns where $k_{ij} = 1$ for some i. This can be computed by subtracting the number of monochromatic columns where $k_{ij} \neq 1$ for all i from the total number of monochromatic columns. Returning to the proof of Lemma 1, a column j was monochromatic if and only if $k_{ij} \equiv j - a_{1j}s + \rho_i s (\bmod 3)$ for all i. Now we add the extra condition that $k_{ij} \neq 1$ for some i. Defining β_j and γ_{ij} as in Lemma 1, for $j \in \{1,\ldots,P\}$, j is a monochromatic column where for all i, $k_{ij} \neq 1$ if and only if for all i, $k_{ij} \neq 1$ and $k_{ij} \equiv \gamma_{ij} (\bmod 3)$ meaning setting $\alpha_{ij} = \frac{(k_{ij}-\gamma_{ij})}{3}$ then $2 \leq 3\alpha_{ij} + \gamma_{ij} \leq p_i$ or, equivalently, $\left\lceil \frac{2-\gamma_{ij}}{3} \right\rceil \leq \alpha_{ij} \leq \left\lfloor \frac{pi-\gamma_{ij}}{3} \right\rfloor = \frac{p_i-s}{3} - \left\lceil \frac{\gamma_{ij}-s}{3} \right\rceil$. For a fixed γ_{ij} there are

$$\begin{aligned} \frac{p_i-s}{3} + \left\lfloor \frac{s-\gamma_{ij}}{3} \right\rfloor + \left\lfloor \frac{\gamma_{ij}-2}{3} \right\rfloor + 1 &= \frac{p_i-s}{3} + \left\lfloor \frac{s-\gamma_{ij}}{3} \right\rfloor + \left\lfloor \frac{\gamma_{ij}+1}{3} \right\rfloor \\ &= \frac{p_i-s}{3} + \left\lfloor \frac{s-((\beta_j+\rho_i s) \bmod 3)}{3} \right\rfloor + \left\lfloor \frac{((\beta_j+\rho_i s) \bmod 3)+1}{3} \right\rfloor \end{aligned}$$

solutions for α_{ij} and hence for k_{ij}. Summing over the three possibilities for β_j , multiplying the numbers of solutions for each row, and multiplying the sum by 3, we obtain the desired expression for the count of monochromatic columns j

such that for all i, $k_{ij} \neq 1$.

$$\begin{aligned} N'(A) &= 3\sum_{\beta=0}^{2}\left(\prod_{i=1}^{t_1}\frac{p_i-s}{3}+\left\lfloor\frac{s-(\beta \bmod 3)}{3}\right\rfloor+\left\lfloor\frac{(\beta \bmod 3)+1}{3}\right\rfloor\right. \\ &\qquad \prod_{i=t_1+1}^{t_2}\frac{p_i-s}{3}+\left\lfloor\frac{s-((\beta+s) \bmod 3)}{3}\right\rfloor+\left\lfloor\frac{((\beta+s) \bmod 3)+1}{3}\right\rfloor \\ &\qquad \left.\prod_{i=t_2+1}^{n}\frac{p_i-s}{3}+\left\lfloor\frac{s-((\beta+2s) \bmod 3)}{3}\right\rfloor+\left\lfloor\frac{((\beta+2s) \bmod 3)+1}{3}\right\rfloor\right) \\ &= 3\sum_{\beta=0}^{2}\prod_{\rho=0}^{2}\prod_{i=t_\rho+1}^{t_{\rho+1}}\left(\frac{p_i-s}{3}+\left\lfloor\frac{s-((\beta+\rho s) \bmod 3)}{3}\right\rfloor+\left\lfloor\frac{((\beta+\rho s) \bmod 3)+1}{3}\right\rfloor\right). \end{aligned}$$

Then

$$B(A) = N(A) - N'(A).$$

Now, consider an arbitrary monochromatic block of width q consisting of columns $j, ..., j+q-1$. Then for $1 \leq i \leq n$, $1 \leq k_{i,j+q-1} = k_{i,j} + q - 1 \leq p_i$ which implies $1 \leq k_{ij} \leq p_i - q + 1$. Hence, q cannot exceed $p = \min(p_1, ..., p_n)$. So, j is the first column of a monochromatic block of width q if and only if j is the first column of a monochromatic block and $k_{ij} = p_i - q + 1$ for some i if and only if j is a monochromatic column and for some i_1, i_2, $k_{i_1 j} = 1$ and $k_{i_2 j} = p_i - q + 1$.

Consider the case when $q = p = \min(p_1, ..., p_n)$. Assume $p_r = p$.

$$\begin{aligned} a_{rj} &= \left(\tfrac{j-k_{ij}}{p_i}+\rho_i\right) \bmod 3 \\ k_{ij} &\equiv j - a_{rj}p_i + \rho_i p_i \pmod 3 \\ &\equiv j - a_{rj}s + \rho_i s \pmod 3 \\ &\equiv j - a_{rj}s + \rho_i s - \rho_r s + \rho_r s \pmod 3 \\ &\equiv k_{rj} - \rho_r s + \rho_i s \pmod 3 \\ &\equiv 1 - \rho_r s + \rho_i s \pmod 3 \end{aligned}$$

noting that $1 \leq k_{rj} \leq p_r - p + 1 = 1$. Let $\gamma_{ij} = (1 - \rho_r s + \rho_i s) \bmod 3$. Then j is the first column of a monochromatic block of width p if and only if $1 \leq k_{ij} \leq p_i - p + 1$ and $k_i \equiv \gamma_{ij} \pmod 3$ meaning setting $\alpha_{ij} = \frac{(k_{ij}-\gamma_{ij})}{3}$, then $1 \leq 3\alpha_{ij} + \gamma_{ij} \leq p_i - p + 1$ or, equivalently, $\left\lceil\frac{1-\gamma_{ij}}{3}\right\rceil \leq \alpha_{ij} \leq \left\lfloor\frac{p_i-p+1-\gamma_{ij}}{3}\right\rfloor = \frac{p_i-p}{3} + \left\lfloor\frac{1-\gamma_{ij}}{3}\right\rfloor$.

There are

$$\begin{aligned} \tfrac{p_i-p}{3}+\left\lfloor\tfrac{1-\gamma_{ij}}{3}\right\rfloor+\left\lfloor\tfrac{\gamma_{ij}-1}{3}\right\rfloor+1 &= \tfrac{p_i-p}{3}+\left\lfloor\tfrac{1-\gamma_{ij}}{3}\right\rfloor+\left\lfloor\tfrac{\gamma_{ij}+2}{3}\right\rfloor) \\ &= \tfrac{p_i-p}{3}+\left\lfloor\tfrac{1-((1-\rho_r s+\rho_i s) \bmod 3)}{3}\right\rfloor+\left\lfloor\tfrac{((1-\rho_r s+\rho_i s) \bmod 3)+2}{3}\right\rfloor) \end{aligned}$$

solutions for α_{ij} and hence for k_{ij}. Multiplying the numbers of solutions for each row, and multiplying the sum by 3, we obtain the desired expression for the count of monochromatic blocks of width p.

$$B_p(3;p_1,p_2,\ldots,p_n;t_1,t_2) = 3\left(\prod_{i=1}^{t_1}\frac{p_i-p}{3}+\left\lfloor\frac{1-((1-\rho_r s) \bmod 3)}{3}\right\rfloor+\left\lfloor\frac{((1-\rho_r s) \bmod 3)+2}{3}\right\rfloor\right.$$
$$\prod_{i=t_1+1}^{t_2}\frac{p_i-p}{3}+\left\lfloor\frac{1-((1-\rho_r s+s) \bmod 3)}{3}\right\rfloor+\left\lfloor\frac{((1-\rho_r s+s) \bmod 3)+2}{3}\right\rfloor$$
$$\left.\prod_{i=t_2+1}^{n}\frac{p_i-p}{3}+\left\lfloor\frac{1-((1-\rho_r s+2s) \bmod 3)}{3}\right\rfloor+\left\lfloor\frac{((1-\rho_r s+2s) \bmod 3)+2}{3}\right\rfloor\right)$$
$$= 3\prod_{\rho=0}^{2}\prod_{i=t_{2\rho}}^{t_{2\rho+1}}\frac{p_i-s}{3}+\left\lfloor\frac{1-((1-\rho_r s+\rho s) \bmod 3)}{3}\right\rfloor+\left\lfloor\frac{((1-\rho_r s+\rho s) \bmod 3)+2}{3}\right\rfloor.$$

Noting that

$$\left\lfloor\frac{1-((1-\rho_r s+\rho s) \bmod 3)}{3}\right\rfloor+\left\lfloor\frac{((1-\rho_r s+\rho s) \bmod 3)+2}{3}\right\rfloor = \begin{cases} 1 \text{ for } \rho_r=\rho \\ 0 \text{ for } \rho_r\neq\rho \end{cases}$$

then

$$B_p(3;p_1,p_2,\ldots,p_n;0,1,2;t_1,t_2) = \begin{cases} 3\prod_{i=1}^{t_1}(q_i+1)\prod_{i=t_1+1}^{t_2}(q_i)\prod_{i=t_2+1}^{n}(q_i) & \text{for } p_r\in\{p_1,\ldots,p_{t_1}\} \\ 3\prod_{i=1}^{t_1}(q_i)\prod_{i=t_1+1}^{t_2}(q_i+1)\prod_{i=t_2+1}^{n}(q_i) & \text{for } p_r\in\{p_{t_1},\ldots,p_{t_2}\} \\ 3\prod_{i=1}^{t_1}(q_i)\prod_{i=t_1+1}^{t_2}(q_i)\prod_{i=t_2+1}^{n}(q_i+1) & \text{for } p_r\in\{p_{t_2},\ldots,p_n\} \end{cases}$$

where $q_i = \frac{p_i-p}{3}$. ■

5 Counting Monochromatic Columns in m number of Colors

Next, we discuss the Monochromatic Column problem for arbitrary m when all the p_i are pairwise coprime. The arguments given in proof of Lemma 1 can be suitably adapted in this case. We give the statement without proof. This improves the result of Jain and Nagpaul [2] as they assume the additional condition that all rows start with the same color.

Lemma 6 *Assume $p_1,\ldots,p_n$ are coprime, $p_i \equiv s (\bmod m)$ for $1\le i\le n$ and fixed $s\in\{1,2,\ldots,m\}$ and $t_0,t_1,\ldots,t_m$ satisfy $0=t_0\le t_1\le t_2\le\cdots\le t_m$. Then*

$$N(m;p_1,p_2,\ldots,p_n;0,1,2,\ldots,m-1;t_1,t_2,\ldots,t_{m-1}) = m\sum_{\beta=0}^{m-1}\prod_{\rho=0}^{m-1}\prod_{i=t_\rho+1}^{t_{\rho+1}}\left(\frac{p_i-s}{m}+G_{\beta,\rho,s}\right)$$

where $G_{\beta,\rho,s} = \left\lfloor\frac{s-((\beta+\rho s) \bmod m)}{m}\right\rfloor+\left\lfloor\frac{((\beta+\rho s) \bmod m)+m-1}{m}\right\rfloor$.

Lemma 3 can be also be generalized to a more general case, namely when m is an arbitrary prime. Again, this generalization is straight forward so we only give the statement here.

Lemma 7 *For a prime m and $n \geq 2$ assume $p_1, \ldots, p_n$ are coprime, $m \mid p_n$, $\rho_n \in \{0, 1, \ldots, m-1\}$, and $t_0, t_1, \ldots, t_{(m-1)m}$ satisfy $0 = t_0 \leq t_1 \leq \cdots \leq t_{(m-1)m}$. Then*

$$N(m; p_1, ..., p_n; 0, 1, 2, \ldots, 0, 1, 2, \rho_n; t_1, \ldots, t_{(m-1)m}) =$$
$$\frac{p_n}{m} N(m; p_1, ..., p_{n-1}; 0, 1, 2, \ldots, 0, 1, 2; t_1, \ldots, t_{(m-1)m-1}).$$

6 Acknowledgement

The authors would like to thank the referee for helpful suggestions that have significantly improved the presentation of the paper. The suggestion of calling the problem the Monochromatic Column Problem was given by the referee which we gratefully acknowledge.

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