



Note

On multicolour noncomplete Ramsey graphs of star graphs

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Received 27 January 2006; received in revised form 7 October 2007; accepted 23 October 2007

Abstract

Given graphs G, G_1, \dots, G_k , where $k \geq 2$, the notation

$$G \rightarrow (G_1, G_2, \dots, G_k)$$

denotes that every factorization $F_1 \oplus F_2 \oplus \dots \oplus F_k$ of G implies $G_i \subseteq F_i$ for *at least one* i , $1 \leq i \leq k$. We characterize G for which

$$G \rightarrow (\mathcal{K}(1, n_1), \mathcal{K}(1, n_2), \dots, \mathcal{K}(1, n_k))$$

and derive some consequences from this. In particular, this gives the value of the graph Ramsey number $\mathcal{R}(\mathcal{K}(1, n_1), \mathcal{K}(1, n_2), \dots, \mathcal{K}(1, n_k))$.

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Keywords: Factorization; r -factor; r -factorable; Edge chromatic number; k -regularization

1. Introduction

Noncomplete Ramsey Theory concerns itself with the factorization of noncomplete graphs. More specifically, given graphs G_1, G_2, \dots, G_k , where $k \geq 2$, we use the standard notation

$$G \rightarrow (G_1, G_2, \dots, G_k) \tag{1}$$

to mean that for every factorization

$$G = F_1 \oplus F_2 \oplus \dots \oplus F_k, \tag{2}$$

we have $G_i \subseteq F_i$ for *at least one* i , $1 \leq i \leq k$. Recall that a factor F is a spanning subgraph of G , and (2) means that the edges of each factor F_i partition the edges of G . Equivalently, if we k -colour the edges of G , the edges of F_i form

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the k colour classes. The natural problem in noncomplete Ramsey Theory is to determine all such G for which (1) holds. Given that it is often difficult to achieve this, it is often the case that one looks at various necessary conditions satisfied by such G , such as giving bounds on its order, size, minimum or maximum degree, chromatic number and clique number. A related problem is the determination of all G for which (1) holds but such that

$$G \setminus e \not\rightarrow (G_1, G_2, \dots, G_k)$$

for all $e \in E(G)$. Such graphs G are called (G_1, G_2, \dots, G_k) -minimal.

Since the general multicolour problem appears quite difficult even for complete graphs, most work has centered around the case $k = 2$. In what follows, we use the standard notation \mathcal{K}_n for the complete graph of order n , and $\mathcal{K}(m, n)$ for the complete bipartite graph with partite sets of orders m, n . Various necessary conditions for the case $G \rightarrow (\mathcal{K}_m, \mathcal{K}_n)$ are known [2,5,6], as is a characterization for G for which $G \rightarrow (\mathcal{K}(1, n), \mathcal{K}(1, n))$. The purpose of this article is to explore those graphs G for which

$$G \rightarrow (\mathcal{K}(1, n_1), \mathcal{K}(1, n_2), \dots, \mathcal{K}(1, n_k)). \tag{3}$$

In view of the fact that

$$G \rightarrow (G_1, G_2, \dots, G_k) \quad \text{if and only if} \quad C \rightarrow (G_1, G_2, \dots, G_k)$$

for some component C of G , provided each G_i is connected, we restrict our attention to connected graphs G .

2. Preliminaries

Since factorization plays a crucial part in the investigation of graphs G for which (1) holds, we first recall some key results. For proofs, we refer the reader to standard texts like [1,11]. We recall that a k -factor of a graph is a k -regular spanning subgraph.

Theorem T1 (Tutte [10]). *A nontrivial graph G has a 1-factor if and only if for every proper subset S of vertices of G , the number of odd components of $G \setminus S$ does not exceed $|S|$.*

Theorem P (Petersen [8]). *A nonempty graph G is 2-factorable if and only if G is $2n$ -regular for some $n \geq 1$.*

An immediate and useful consequence is that a $2n$ -regular graph has a $2m$ -factor for each $m < n$. Another useful result is the following.

Lemma A. *For every $n \geq 1$, \mathcal{K}_{2n} is 1-factorable.*

The concept of a k -factor has a generalization in the following sense. Given a graph G and a function $f : V(G) \rightarrow \mathbb{N} \cup \{0\}$, G is said to have an f -factor provided it has a subgraph H such that $\deg_H v = f(v)$ for each $v \in V(G)$. Tutte [9] gave a necessary and sufficient condition for a graph to have an f -factor, relating it to checking whether a related graph (G, f) has a 1-factor. The construction of (G, f) is as follows:

Corresponding to each vertex v of G are complete bigraphs $\mathcal{K}(d(v), e(v))$, with partite sets $A(v)$ of size $d(v) = \deg v$ and $B(v)$ of size $e(v) = \deg v - f(v)$. Corresponding to each edge uv of G , join one vertex of $A(u)$ with one vertex of $A(v)$.

Theorem T2 (Tutte [9]). *A graph G has an f -factor if and only if the graph (G, f) has a 1-factor.*

The following result, due to U.S.R. Murty, is closely connected and central to our paper.

Theorem M (Murty). *Let G be a connected graph and n a positive integer. Then $G \rightarrow (\mathcal{K}(1, n), \mathcal{K}(1, n))$ if and only if*

- (a) $\Delta(G) \geq 2n - 1$, or
- (b) n is even and G is a $(2n - 2)$ -regular graph of odd order.

One class of graphs for which the Ramsey numbers are exactly known is the set of graphs each of which is a star graph. Given graphs G_1, G_2, \dots, G_k , where $k \geq 2$, the graph Ramsey number $\mathcal{R}(G_1, G_2, \dots, G_k)$ is the least

Lemma 2. Let n_1, n_2, \dots, n_k be positive integers. If $\Delta(G) = N$, then

$$G \rightarrow (\mathcal{K}(1, n_1), \mathcal{K}(1, n_2), \dots, \mathcal{K}(1, n_k))$$

if and only if

$$G^* \rightarrow (\mathcal{K}(1, n_1), \mathcal{K}(1, n_2), \dots, \mathcal{K}(1, n_k)).$$

Proof. Suppose G^* satisfies (n_1, n_2, \dots, n_k) and G does not. Then $G = F_1 \oplus F_2 \oplus \dots \oplus F_k$, with $\Delta(F_i) \leq n_i - 1$ for $1 \leq i \leq k$. However, $\Delta(G) = N$ forces $\Delta(F_i) = n_i - 1$ for $1 \leq i \leq k$. The edge sum of the Δ -regularization of these factors is then N -regular, and hence it is the Δ -regularization of G . But this contradicts our assumption that G^* satisfies (n_1, n_2, \dots, n_k) . The converse is trivial. \square

Lemma 3. If G is r -regular, then G_{r+1}^* is 1-factorable.

Proof. Let G be an r -regular graph. Then, by Theorem V, G is $r + 1$ edge-colourable. The construction of G_{r+1}^* involves making two copies of G and joining identical vertices. We use identical colours for edges in the two copies of G , using $r + 1$ colours. Moreover, since each vertex v has degree r in G , only r colours are used for colouring the edges incident with v in each copy. Thus, there is a colour free for the edge joining identical vertices in the two copies. This proves that G_{r+1}^* is also $r + 1$ edge-colourable. Then the spanning subgraphs with edges from each of the $r + 1$ colour classes are 1-factors of G_{r+1}^* . \square

Lemma 4. Let n_1, n_2, \dots, n_k be positive integers, and let G be a connected graph. The following are equivalent:

- (a) $G \rightarrow (\mathcal{K}(1, n_1), \mathcal{K}(1, n_2), \dots, \mathcal{K}(1, n_k)) \implies \Delta(G) \geq N$ holds for every choice of positive integers n_1, n_2, \dots, n_k .
- (b) G is r -regular $\implies G_{r+1}^*$ is 1-factorable.

Proof. (a) \implies (b): Let G be an r -regular graph. Then

$$G \not\rightarrow \underbrace{(\mathcal{K}(1, 2), \mathcal{K}(1, 2), \dots, \mathcal{K}(1, 2))}_{r+1 \text{ terms}},$$

since $\Delta(G) = r = N - 1$ and part (a) holds. Thus, there is a factorization

$$G = F_1 \oplus F_2 \oplus \dots \oplus F_{r+1},$$

with $\Delta(F_i) = 1$ for each i . Since G is r -regular, corresponding to each vertex v of G , there is a factor $F_{i(v)}$ such that $\deg v$ equals 1 in each factor except in $F_{i(v)}$ where $\deg v = 0$.

Fix $i, 1 \leq i \leq r + 1$. Then the subgraph H_i of G_{r+1}^* consisting of two copies of F_i together with the edges joining those identical vertices of degree 0 in F_i is a 1-factor of G_{r+1}^* . Thus, G_{r+1}^* is 1-factorable.

(b) \implies (a): Suppose G satisfies (n_1, n_2, \dots, n_k) and $\Delta(G) < N$. Since G_{N-1}^* is an $(N - 1)$ -regular graph, by part (b), G_N^* is 1-factorable. Thus, we can write

$$\begin{aligned} G_N^* &= F_1 \oplus F_2 \oplus \dots \oplus F_N \\ &= H_1 \oplus H_2 \oplus \dots \oplus H_k, \end{aligned}$$

where each factor F_i is 1-regular, H_1 equals $F_1 \oplus \dots \oplus F_{n_1-1}$, H_2 equals the edge sum of the next $(n_2 - 1)F_i$'s, and so on, so that each factor H_i is $(n_i - 1)$ -regular. But then

$$G = (H_1 \cap G) \oplus (H_2 \cap G) \oplus \dots \oplus (H_k \cap G)$$

implies that G does not satisfy (n_1, n_2, \dots, n_k) . This contradiction proves $\Delta(G) \geq N$. \square

Theorem 1. Let G be a connected graph such that

$$G \rightarrow (\mathcal{K}(1, n_1), \mathcal{K}(1, n_2), \dots, \mathcal{K}(1, n_k)).$$

Then $\Delta(G) \geq N$.

Proof. This is a direct consequence of Lemmas 3 and 4. □

Theorem 2. Let G be a connected graph, and let n_1, n_2, \dots, n_k be positive integers of which e are even. Let G^* be the Δ -regularization of G , as defined above. Then

$$G \rightarrow (\mathcal{K}(1, n_1), \mathcal{K}(1, n_2), \dots, \mathcal{K}(1, n_k))$$

if and only if

- (a) $\Delta(G) \geq N + 1$, or
- (b) G is N -regular, of odd order and e is even and non-zero, or
- (c) G is N -regular, of even order, at least one n_i is even, and G does not have an $(n_i - 1)$ -factor for at least one even n_i , or
- (d) G is not N -regular, $\Delta(G) = N$ and $G^* \rightarrow (\mathcal{K}(1, n_1), \mathcal{K}(1, n_2), \dots, \mathcal{K}(1, n_k))$.

Proof. Suppose first that at least one of the conditions is met.

Condition (a) implies G satisfies (n_1, n_2, \dots, n_k) by the Pigeonhole Principle. Suppose condition (b) holds and $G = F_1 \oplus F_2 \oplus \dots \oplus F_k$, with $\Delta(F_i) \leq n_i - 1$ for $1 \leq i \leq k$. The regularity of G forces each F_i to be $(n_i - 1)$ -regular. But each F_i is of odd order and at least one $n_i - 1$ is odd, and this is a contradiction.

Suppose next that condition (c) is met. Arguing as in the previous case, we observe that if G did not satisfy (n_1, n_2, \dots, n_k) , it must have an $(n_i - 1)$ -factor for each i . But G does not have such a factor for even n_i , and so this is not the case.

Finally, suppose condition (d) holds. By Lemma 2, G satisfies (n_1, n_2, \dots, n_k) . This completes the sufficiency of each of the four conditions.

Conversely, suppose G satisfies (n_1, n_2, \dots, n_k) . If $\Delta(G) \geq N + 1$, there is nothing to prove; suppose $\Delta(G) \leq N$.

If G is N -regular and of odd order, then N must be even, so that e must be even. If $e = 0$, each $n_i - 1$ is even and since G is 2-factorable by Theorem P, G has an $(n_i - 1)$ -factor for each i . This contradicts the assumption that G satisfies (n_1, n_2, \dots, n_k) . Thus, in this case, e must be non-zero.

Suppose that G is N -regular and of even order. If G has an $(n_i - 1)$ -factor for each even n_i , then the graph obtained from G by removing each of these factors is regular of even degree, and hence 2-factorable, and so has $(n_i - 1)$ -factors for odd n_i as well. But then G does not satisfy (n_1, n_2, \dots, n_k) , and this contradiction implies that G does not have $(n_i - 1)$ -factors for at least one even n_i .

Finally, suppose G is not N -regular. Since G satisfies (n_1, n_2, \dots, n_k) , $\Delta(G) = N$ by Theorem 1, and clearly G^* satisfies (n_1, n_2, \dots, n_k) as well. This completes the characterization. □

The characterization of G that satisfies (n_1, n_2, \dots, n_k) as given by Theorem 2 makes it easy to determine the Ramsey numbers and the bipartite Ramsey numbers of star graphs. Ramsey numbers of star graphs were determined by Burr and Roberts (Theorem BR in Section 2). However, our proof derives their result as a consequence of a more general result, and is not restricted to determining only complete graphs that satisfies (n_1, n_2, \dots, n_k) .

Corollary 1 (Burr and Roberts [3]). Let n_1, n_2, \dots, n_k be positive integers, e of which are even. Then

$$\mathcal{R}(\mathcal{K}(1, n_1), \mathcal{K}(1, n_2), \dots, \mathcal{K}(1, n_k)) = \begin{cases} N + 1 & \text{if } e \text{ is even and positive;} \\ N + 2 & \text{otherwise,} \end{cases}$$

where $N = \sum_{i=1}^k (n_i - 1)$.

Proof. This is a direct consequence of Theorem 2. Observe that \mathcal{K}_{N+2} satisfies (n_1, n_2, \dots, n_k) by condition (a). To complete the proof, we need to show that \mathcal{K}_{N+1} satisfies (n_1, n_2, \dots, n_k) if and only if e even and non-zero. If e is even and non-zero, condition (b) applies to \mathcal{K}_{N+1} . Conversely, suppose \mathcal{K}_{N+1} satisfies (n_1, n_2, \dots, n_k) . If N is even, by condition (b), e is even and non-zero. If N is odd, by condition (c), \mathcal{K}_{N+1} does not have an $(n_i - 1)$ -factor for at least one even n_i , which contradicts Lemma A. □

Corollary 2. Let n_1, n_2, \dots, n_k be positive integers, and let $N = (n_1 - 1) + (n_2 - 1) + \dots + (n_k - 1)$. Then

$$\mathcal{K}(p, p) \rightarrow (\mathcal{K}(1, n_1), \mathcal{K}(1, n_2), \dots, \mathcal{K}(1, n_k))$$

if and only if $p \geq N + 1$.

Proof. Suppose $\mathcal{K}(p, p) \rightarrow (\mathcal{K}(1, n_1), \mathcal{K}(1, n_2), \dots, \mathcal{K}(1, n_k))$. Since $\mathcal{K}(p, p)$ is 1-factorable, $\mathcal{K}(N, N)$ has an $(n_i - 1)$ -factor for each i , $1 \leq i \leq k$, so that $\mathcal{K}(N, N)$ does not satisfy (n_1, n_2, \dots, n_k) . Therefore, $p > N$. Conversely, $\mathcal{K}(N + 1, N + 1)$, and hence $\mathcal{K}(p, p)$ for each $p \geq N + 1$, satisfies (n_1, n_2, \dots, n_k) because of the Pigeonhole Principle. \square

Acknowledgements

The authors are grateful to the referees for their careful reading, their helpful comments and for a reference.

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