ON Σ-V RINGS

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Abstract. We discuss various properties of a ring over which each simple module is Σ-injective.

We shall consider associative rings with identity. Our modules will be unital right modules unless stated otherwise. The class of right V- rings was introduced by Villamayor [20]. A ring R is called a right V- ring if each simple right R-module is injective. It is a well-known unpublished result due to Kaplansky that a commutative ring is von Neumann regular if and only if it is a V- ring. The class of V- rings and, in particular, V- domains has been studied by many authors including Baccella ([1], [2]), Cozzens [5], Faith ([7], [8]), Fisher [10], Goodearl [14], Goursaud [12], Huynh et al [16], Kaplansky, Jain et al [17], Michler et al. [20], Osofsky [21], Page [22], Tyukavkin [24] and Tuganbaev [25]. Goursaud and Valette generalized the notion of V- ring and called a ring R to be a right Σ-V ring if each simple right R-module is Σ-injective [13]. Recall that a module M is called Σ-injective if every direct sum of copies of M is injective [6]. After defining these rings, Goursaud and Valette studied only the particular class of von Neumann regular right Σ-V rings [13]. Their paper [13] was published in 1975 but surprisingly in past 34 years no other paper except that of Baccella [1] even mentions Σ-V rings and this class of rings seems to have been forgotten. The present paper attempts to bring this interesting class of rings in light again for further research.

Clearly, every right Σ-V ring is a right V- ring, however there are examples of right V- rings which are not right Σ-V rings.

1. Finiteness Properties of Σ-V rings

It is well-known that a ring R is right noetherian if and only if each injective right R-module is Σ-injective [3]. Since in a right Σ-V ring we require each simple module to be Σ-injective, we expect some kind of finiteness in such rings. In this section we will establish finiteness properties of Σ-V rings. We begin with a useful observation about Σ-V rings. First, recall that a ring R is called right $q.f.d.$ relative to a module M if no cyclic right R-module contains an infinite direct sum of modules isomorphic to submodules of M.

Lemma 1. *Let R be a right Σ-V ring and let S be any simple right R-module. Then R is right $q.f.d.$ relative to S.*

Proof. It follows from the fact that if a right R-module M is Σ-injective, then R must be right $q.f.d.$ relative to M (see [15]). $\square$

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The proof of the next lemma is inherent in [4] where similar results have been obtained for class of rings over which each essential extension of a direct sum of simple modules is a direct sum of quasi-injective modules. But as we will see here that those techniques may be applied in a more general situation and much more general results may be obtained. Since these results have been stated in their most general form only in the author's Ph.D. dissertation [23], instead of leaving the reader wondering how to adapt the proof given in [4], we will give the proof here for the sake of completeness.

It is not difficult to observe that a ring R with finite right uniform dimension must be directly-finite. The standard technique to prove this is that we assume to the contrary that R is not directly-finite. This gives rise to an infinite set of orthogonal idempotents in R yielding an infinite direct sum of right ideals contained in R, a contradiction to the finite right uniform dimension of R. In the next result we generalize this argument.

Lemma 2. *Let R be a ring which is right q.f.d. relative to every simple R-module. Then*

(a) R must be directly-finite.

(b) If, in addition, R is right non-singular then the maximal right ring of quotients of R, $Q^r_{\max}(R)$ is a finite direct product of matrix rings over abelian regular right self-injective rings.

Proof. (a) Assume to the contrary that R is not directly finite. Then, there exist $x, y \in R$ such that $xy = 1$ and $yx \neq 1$. Set $e_{ij} = y^{i-1}x^{j-1} - y^i x^j$ for all $(i, j) \in \mathbb{N} \times \mathbb{N}$. Then $\{e_{ij} : (i, j) \in \mathbb{N} \times \mathbb{N}\}$ is an infinite set of nonzero matrix units. For each $n \in \mathbb{N}$, we will produce cyclic submodules $C_{n,i}$ of R_R, where $C_{n,i} \cong C_{n,j}$ for all $i, j = 2, 3, ..., n$. We produce these cyclic submodules by induction.

Let $n > 1$. Let us denote by Q, the maximal right ring of quotients of R. Since $R \subseteq_e Q$, there exists a nonzero cyclic submodule $C_{n,1}$ of R_R such that $C_{n,1} \subseteq e_{n^2,n^2}Q \cap R$. Now we choose $x_2 \in e_{n^2+1,n^2}C_{n,1} \cap R$. Then $x_2 = e_{n^2+1,n^2}x_1$, where $x_1 \in C_{n,1}$. Denote $C_{n,2} = x_2R$ and redefine $C_{n,1}$ by setting $C_{n,1} = x_1R$. Define the module homomorphism $\varphi : C_{n,1} \longrightarrow C_{n,2}$ by $\varphi(x) = e_{n^2+1,n^2}x$. Clearly, φ is an isomorphism, and so $C_{n,1} \cong C_{n,2}$. Suppose now that we have defined cyclic submodules $C_{n,1} \cong C_{n,2} \cong ... \cong C_{n,j-1}$ in R, where $C_{n,i} = x_iR$, $i = 1, 2, ..., j-1$. Next, we choose x_j such that $x_j \in e_{n^2+j-1,n^2+j-2}C_{n,j-1} \cap R$ and write $x_j = e_{n^2+j-1,n^2+j-2}x_{j-1}r_{j-1}$ where $r_{j-1} \in R$. Let $x'_{j-1} = x_{j-1}r_{j-1}$, and set $C_{n,j} = x_jR$. Now redefine $C_{n,j-1} = x'_{j-1}R$ (which is contained in the previously constructed $C_{n,j-1}$). Then $C_{n,j-1} \cong C_{n,j}$ under the isomorphism that sends $x \in C_{n,j-1}$ to $e_{n^2+j-1,n^2+j-2}x$. We redefine preceding $C_{n,1}, C_{n,2}, ..., C_{n,j-2}$ accordingly so that they all remain isomorphic to each other and to $C_{n,j-1}$. Note that the family $\{C_{n,i}\colon\ n = 2, 3, ...,\ \ i = 1, 2, ..., n\}$ is independent. By our construction, $C_{n,i} \cong C_{n,j}$ for all $n = 2, 3, ...$ and $1 \leq i, j \leq n$. Therefore, there exist maximal submodules $M_{n,i}$ of $C_{n,i}$, $n = 2, 3, ...$ and $1 \leq i \leq n$, such that $C_{n,i}/M_{n,i} \cong C_{n,j}/M_{n,j}$ for all n, i, j. Set $M = \oplus_{n,i}M_{n,i}$, and $S_n = C_{n,1}/M_{n,1}$. Clearly, we have $\frac{R}{M} \supset \frac{\oplus C_{n,i}}{\oplus M_{n,i}} \cong \frac{C_{2,1}}{M_{2,1}} \times \frac{C_{2,2}}{M_{2,2}} \times \frac{C_{3,1}}{M_{3,1}} \times ....$Thus, R/M is a cyclic right R-module that contains an infinite direct sum of modules each isomorphic to simple module S_n. This yields a contradiction to our hypothesis. Therefore, R must be directly finite.

(b) We know that $Q = Q^r_{\max}(R)$ is a von Neumann regular right self-injective ring. By part (a), the ring R is directly finite and hence so is the ring Q. Hence by the type theory of von Neumann regular right self-injective rings, $Q = Q_1 \times Q_2$ where Q_1 is of Type I_f and Q_2 is of Type II_f (see [14], Theorem 10.22). Now we claim that Q must be of type I_f, that is, $Q_2 = 0$. Assume to the contrary that $Q_2 \neq 0$. Then by ([14], Proposition 10.28), there exists an idempotent $e_3^{'} \in Q_2$ such that $(Q_2)_{Q_2} \cong 3(e_3^{'}Q_2)$. Therefore, $Q_2 = e_1Q_2 \oplus e_2Q_2 \oplus e_3Q_2$ where $e_1, e_2, e_3 \in Q_2 \subseteq Q$ are nonzero orthogonal idempotents such that their sum is the identity of the ring Q_2. Clearly, $e_iQ = e_iQ_2$, and $e_jQ_2 = e_jQ$ for all $1 \leq i, j \leq 3$ and so $e_iQ \cong e_jQ$ for all $1 \leq i, j \leq 3$. Therefore, there exist nonzero cyclic submodules $C_{2i} \subseteq e_iQ \cap R$, $i = 1, 2$, such that $C_{21} \cong C_{22}$. By ([14], Corollary 10.9), $e_3Q_2e_3 = \mathrm{End}(e_3Q_2)$ is of Type II_f and so as above there exist nonzero orthogonal idempotents $f_1, f_2, f_3, f_4 \in e_3Q_2e_3$ such that $f_i(e_3Q_2e_3) \cong f_j(e_3Q_2e_3)$ for all $1 \leq i, j \leq 4$. Hence, $f_i(e_3Qe_3) \cong f_j(e_3Qe_3)$ for all i, j. By ([19], Proposition 21.20), there exist $a \in f_i(e_3Qe_3)f_j$ and $b \in f_j(e_3Qe_3)f_i$ such that $f_i = ab$ and $f_j = ba$. Then for all i, j, we have $f_iQ \cong f_jQ$ under the mapping which sends f_ix to bf_ix for each $x \in Q$. Furthermore, there exist nonzero cyclic submodule $C_{3i} \subseteq f_iQ \cap R$, $i = 1, 2, 3$ such that $C_{3i} \cong C_{3j}$ for all $1 \leq i, j \leq 3$. Continuing in this fashion, we construct an independent family $\{C_{ij} : i = 2, 3, ...; 1 \leq j \leq i\}$ of nonzero cyclic submodules of R such that $C_{ij} \cong C_{ik}$ for all $1 \leq j, k \leq i$; $i = 2, 3, ...$ Therefore, there exist maximal submodules M_{ij} of $C_{ij}, 1 \leq j \leq i$; $i = 2, 3, ...$ such that $C_{ij}/M_{ij} \cong C_{ik}/M_{ik}$ for all i, j, k. Setting $M = \oplus_{i,j}M_{ij}$, and $S_i = C_{i1}/M_{i1}$, we get that the cyclic right module R/M contains an infinite direct sum of modules each isomorphic to S_i, a contradiction to our hypothesis. Therefore, $Q_2 = 0$ and Q is of Type I_f. Hence $Q = \Pi_{i=1}^{\infty} Q_i$ where each Q_i is an $i \times i$ matrix ring over an abelian regular self-injective ring (see [14], Theorem 10.24). Now, we claim that this product must be a finite product. Assume to the contrary that this product is infinite, then for any positive integer n there exists an index $m \geq n$ such that $Q_m \neq 0$. Now, for any fixed k, we have matrix units $\{e^k_{ij} : 1 \leq i, j \leq k\}$ which are $k \times k$ matrices. Thus, we have an infinite family of nonzero matrix units $\{\{e^k_{ij}$: $1 \leq i, j \leq k\}$: $k = 2, 3, ...\} \subseteq Q$. Since $R \subseteq_e Q$, there exists a nonzero cyclic submodule $C_{k,1}$ of R such that $C_{k,1} \subseteq e^k_{1,1}Q \cap R$ and then starting with $C_{k,1}$, we construct an independent family $\{C_{k,i}$: $k = 2, 3, ...,\ \ 1 \leq i \leq k\}$ of cyclic submodules of R such that $C_{k,i} \cong C_{k,j}$ for all k, i, j (exactly as shown in the proof of part (a)). Therefore, there exist maximal submodules $M_{k,i}$ of $C_{k,i}$, $k = 2, 3, ...$ and $1 \leq i \leq k$, such that $C_{k,i}/M_{k,i} \cong C_{k,j}/M_{k,j}$ for all k, i, j. Set $M = \oplus_{k,i}M_{k,i}$, and $S_k = C_{k,1}/M_{k,1}$. Note that the cyclic right module R/M contains an infinite direct sum of modules each isomorphic to S_k, which contradicts to our hypothesis. Therefore, $Q = Q^r_{\max}(R)$ is the direct product of a finite number of matrix rings over abelian regular self-injective rings. □

Theorem 3. *Every right Σ-V ring is directly finite.*

Proof. Let R be a right Σ-V ring. Let S be any simple right R-module then by Lemma 1, R is right $q.f.d.$ relative to S. Therefore, by Lemma 2(a), R must be directly finite. □

Example 4. *A right V- ring need not be directly finite. Let F be a field and V a countable dimensional vector space over F. Let $Q = End_F(V)$. Consider the ring*

$R = Soc(Q) + F(x)$. It may be shown that R is a right V- ring but R is not directly finite.

This is an example of a ring which is a right V- ring but not a right (or left) Σ-V ring.

Theorem 5. *Let R be a right non-singular, right Σ-V ring. Then the maximal right ring of quotients of R, $Q^r_{\max}(R)$ is a finite direct product of matrix rings over abelian regular right self-injective rings.*

Proof. It follows from Lemma 1 and Lemma 2(b). □

As a consequence of above, we have the following.

Corollary 6. *Let R be a right non-singular right Σ-V ring. Then R must have bounded index of nilpotence.*

Proof. By above theorem, the maximal right ring of quotients $Q^r_{\max}(R)$ has bounded index of nilpotence and hence R must have bounded index of nilpotence. □

A right non-singular right V-ring need not have bounded index of nilpotence. The following result of Tyukavkin [24] gives a condition when a von Neumann regular right self-injective right V-ring has bounded index of nilpotence.

Theorem 7. (*Tyukavkin,* [24]) *Let R be a von Neumann regular right self-injective right V-ring such that the dimension of every simple right R-module S over the division ring of endomorphisms of S is less than $2^{2^{\aleph_0}}$. Then R has bounded index of nilpotence.*

2. von Neumann regular right Σ-V rings

The class of von Neumann regular right V-rings has been of considerable importance. As mentioned earlier, Kaplansky was the first to observe that in the case of commutative rings the notions of von Neumann regular rings and V-rings coincide. However, in case noncommutative rings these are two distinct classes of rings. If V_D is an infinite-dimensional vector space over a division ring D, then the ring of linear transformations $R = \mathrm{End}(V_D)$ is a von Neumann regular ring but R is not a left V-ring as V is a simple left R-module which is not injective. The example of Cozzens [5] is a right V-ring which is not von Neumann regular. Goursaud and Valette ([13], Corollaire 2.7) showed that a von Neumann regular ring with artinian primitive factors is both a left and right V-ring. Baccella [1], showed the converse and proved that a right V-ring with artinian primitive factors must be von Neumann regular.

The von Neumann regular right Σ-V rings were studied by Goursaud and Valette [13] and later by Baccella [1]. In this section we will show that their results may be obtained as easy consequences of our results.

Since von Neumann regular rings are right non-singular, we have the following result of Goursaud and Valette as a special case of our Theorem 5.

Lemma 8. [13] *Let R be a von Neumann regular right self-injective right Σ-V ring. Then R is a finite direct product of matrix rings over abelian regular right self-injective rings.*

In the next theorem, we rediscover the characterization of von Neumann regular right Σ-V rings given by Goursaud-Valette [13] and Baccella [1] as a consequence of our Corollary 6.

Theorem 9. ([13], [1]) *For a von Neumann regular ring R, the following are equivalent:*

(a) R is a left Σ-V *ring.*

(b) For any prime ideal P, R/P is artinian.

(c) R is a right Σ-V *ring.*

Proof. $(a) \implies (b)$. Let R be a von Neumann regular right Σ-V ring. By Corollary 6, R has bounded index of nilpotence. Let P be any prime ideal of R. Then R/P is a prime von Neumann regular ring with bounded index of nilpotence. Hence, R/P must be simple artinian (see [14], Theorem 7.9).

$(c) \implies (b)$ is similar.

$(b) \implies (c)$. This follows easily from a standard argument that was first given by Faith in case of V-rings [7] and later used by Goursaud-Valette [13] and Baccella [1] for Σ-V rings. For the sake of completeness, we give the proof of this part again. Let S be a simple right R-module. Let $P = ann^r_R(S)$. By assumption R/P is artinian. Therefore, S is Σ-injective as an R/P-module. Since R is a von Neumann regular ring, R/P is flat as a right R-module. Then by ([14], Lemma 6.17), S is Σ-injective as a right R-module as well. Hence, R is a right Σ-V ring.

The proof of $(b) \implies (a)$ is similar.

□

Remark 10. *The above theorem shows that the class of von Neumann regular* Σ-V *rings is left-right symmetric.*

Example 11. *Let V be an infinite-dimensional vector space over F. Set $Q = \mathrm{End}_F(V)$, $J = \{x \in Q : dim_F(xV) < \infty\}$ and $R = F + J$. Then R is a von Neumann regular right V- ring which is not a left V- ring (see* [14], *Example* 6.19*). This shows that von Neumann regular V rings are not left-right symmetric.*

This example also shows that a von Neumann regular right V-ring need not have all primitive factors artinian.

Proposition 12. *Let R be a right non-singular prime right* Σ-V *ring. Then R is a simple right Goldie ring.*

Proof. By Corollary 6, $Q^r_{\max}(R)$ has bounded index of nilpotence. Thus $Q^r_{\max}(R)$ is a prime von Neumann regular ring with bounded index of nilpotence, hence simple artinian. Therefore, R is a simple right Goldie ring. □

Next, we provide a characterization of right Σ-V rings which follows easily from the new characterization of Σ-injective modules due to Guil Asensio et al [15].

Theorem 13. *A ring R is a right* Σ-V *ring if and only if for each simple right R-module S, each essential extension of $S^{(\aleph_0)}$ is a direct sum of injective modules.*

Proof. It follows from ([15], Corollary 5). □

3. Open Problems

We conclude the paper with a list of open problems which we think are important to better understand the class of right Σ-V rings.

Problem 14. *We have already seen that the von Neumann regular Σ-V rings are left-right symmetric. For which other classes of rings, Σ-V rings are left-right symmetric?*

Problem 15. *Does every exchange right Σ-V ring have bounded index of nilpotence?*

Problem 16. *Characterize a simple right Goldie right Σ-V ring.*

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