On Unit-Central Rings

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Dedicated to S. K. Jain in honor of his 70th birthday.

Abstract. We establish commutativity theorems for certain classes of rings in which every invertible element is central, or, more generally, in which all invertible elements commute with one another. We prove that if R is a *semiexchange ring* (i.e. its factor ring modulo its Jacobson radical is an exchange ring) with all invertible elements central, then R is commutative. We also prove that if R is a semiexchange ring in which all invertible elements commute with one another, and R has no factor ring with two elements, then R is commutative. We offer some examples of noncommutative rings in which all invertible elements commute with one another, or are central. We close with a list of problems for further research.

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1. Introduction

We say that an associative unital ring R is *unit-central* if $\mathrm{U}(R) \subseteq \mathrm{Z}(R)$, i.e. if every invertible element of the ring lies in the center. In various natural situations the unit-central condition implies full commutativity.

It is also of interest to weaken the unit-central condition and consider rings R for which $\mathrm{U}(R)$ is an abelian group. We will refer to such a ring R as having *commuting units*. Rings with commuting units have also been investigated by a number of authors (e.g. see [7], [12], [21], [22]). For a ring that is additively generated by its units (cf. [17], [18], [19], [26], [28]), having commuting units is obviously equivalent to commutativity.

Our main focus in this note will be on unit-central rings and rings with commuting units. A still wider class consists of those rings in which any two nilpotent elements commute with one another. This property proved instrumental in the study of prime rings in [5]. We will consider this condition in Theorem 2.8 below.

We will denote the Jacobson radical of a ring R by $\mathrm{rad}(R)$, the set of nilpotent elements by $\mathfrak{N}(R)$, and the right annihilator of an element a in R by $\mathrm{ann}_r^R(a)$. For any other notation not defined here, we refer the reader to [20].

We record the following construction technique for the classes of rings under consideration.

Proposition 1.1. *Let S be a ring, let M be an (S,S)-bimodule, and define $R = S \oplus M$ as an additive group, with multiplication in R defined by $(s_1, m_1)(s_2, m_2) = (s_1 s_2, s_1 m_2 + m_1 s_2)$.*

(i) *R is unit-central if and only if S is unit-central and $sm = ms$ for all $s \in S$ and $m \in M$.*

(ii) *R has commuting units if and only if S has commuting units and $sm = ms$ for all $s \in \mathrm{U}(S)$ and $m \in M$.*

Proof. Straightforward. □

2. Commutativity theorems

We begin with a basic but useful lemma. Recall that a ring is said to be *abelian* if every idempotent element is central.

Lemma 2.1. *Let R be a ring. Then:*

(i) *If R is unit-central, then $\mathfrak{N}(R) \cup \mathrm{rad}(R) \subseteq \mathrm{Z}(R)$.*

(ii) *If R has commuting units, then for all $a, b \in \mathfrak{N}(R) \cup \mathrm{rad}(R) \cup \mathrm{U}(R)$ we have $ab = ba$.*

(iii) *If R is unit-central, then R is abelian.*

(iv) *If for all $a, b \in \mathfrak{N}(R)$ we have $ab = ba$, then R is Dedekind-finite.*

Proof. Statements (i) and (ii) are straightforward. If $e \in R$ is an idempotent in a unit-central ring, then (i) implies $eR(1-e) = \{0\}$, and (iii) follows. A Dedekind-infinite ring contains an infinite set of matrix units, whence (iv). □

Obviously neither the property of having commuting units nor the unit-central condition is Morita invariant; however, they do pass to corner rings:

Lemma 2.2. *Let R be a ring and $e \in R$ an idempotent. If R is unit-central (resp. a ring with commuting units), then the corner ring eRe is unit-central (resp. a ring with commuting units).*

Proof. Suppose R is unit-central, with $ere \in \mathrm{U}(eRe)$ and $ese \in eRe$. Then $ere + (1-e)$ is contained in $\mathrm{U}(R)$, so it commutes with ese, and hence ere and ese commute.

The proof for the "commuting units" case is analogous. □

Recall that a ring R is called an *exchange ring* if the module R_R satisfies P. Crawley and B. Jónsson's exchange property: given a set I, whenever

$$A = M \oplus N = \bigoplus_{i \in I} A_i \qquad \text{with} \qquad M \cong R$$

in the category of right R-modules, there exist submodules $A_i' \subseteq A_i$ such that

$$A = M \oplus \left(\bigoplus_{i \in I} A_i' \right).$$

R. W. Warfield Jr. showed in [29, Corollary 2] that this property is left-right symmetric. Every ring R that is *semiregular* (i.e. $R/\mathrm{rad}(R)$ is von Neumann regular and idempotents of $R/\mathrm{rad}(R)$ lift to R) is an exchange ring. Also, every *clean ring* (i.e. ring in which every element is the sum of a unit and an idempotent) is an exchange ring. For example, the endomorphism ring of an continuous module is both semiregular and clean (for the latter, see [9]). By [4, Proposition 2.6], every strongly π-regular ring is clean; by [27, Example 2.3], every π-regular ring is an exchange ring. In addition to semiregular and clean rings, the class of exchange rings includes all C^*-algebras of real rank zero and Gromov translation rings of discrete trees over von Neumann regular rings (see [2, Theorem 7.2] and [3, Theorem 2.7]).

Exchange rings can be characterized as those rings for which every pair of comaximal right ideals contain a complementary pair of idempotents, i.e. R is an exchange ring if and only if for each element $a \in R$ there exists an idempotent $e \in R$ such that $e \in aR$ and $1-e \in (1-a)R$. This characterization was independently discovered by K. R. Goodearl and W. K. Nicholson (see [14, p. 167] and [23, Proposition 1.1 and Theorem 2.1]), and it is very useful in practice. For example, the proof by P. Ara, K. C. O'Meara, and F. Perera that Gromov translation rings of discrete trees over von Neumann regular rings are exchange rings in [3] relied crucially on Goodearl and Nicholson's characterization.

A ring R is said to be a *semiexchange ring* if the factor ring $R/\mathrm{rad}(R)$ is an exchange ring. This common generalization of exchange rings and semilocal rings arises naturally: according to [23, Corollary 2.4], a ring is an exchange ring if and only if it is a semiexchange ring and idempotents lift modulo the Jacobson radical. The (apparently rather deep) open problem of the left-right symmetry of the quasi-duo condition has an affirmative answer for the class of semiexchange rings (see [11, Theorem 4.6]). Basic properties of semiexchange rings are developed in [10]. Of course, a semiexchange ring need not be either semilocal or an exchange ring, as can be seen by taking a direct product of a semilocal ring and an exchange ring, or an infinite direct product of semilocal rings.

A ring with commuting units can be both semilocal and an exchange ring without being commutative. On the other hand, a unit-central ring that is either semilocal or an exchange ring must be commutative, by the following theorem.

Theorem 2.3. *Every unit-central semiexchange ring is commutative.*

Proof. Let R be a unit-central semiexchange ring. To prove that R is commutative, by [20, p. 200, Ex. 12.8B] it suffices to show that $x - x^2 \in \mathrm{Z}(R)$ for every $x \in R$.

Fix $x \in R$. The exchange ring $\overline{R} = R/\mathrm{rad}(R)$ is unit-central, therefore abelian, and it is well known that abelian exchange rings are clean [23, Proposition 1.8(2)]. So $x = e + u$ for some $e, u \in R$ such that $\overline{e}$ is an idempotent and $\overline{u}$

a unit of $\overline{R}$. Then $\overline{1-2e} \in \mathrm{U}(\overline{R})$, so $1-2e \in \mathrm{U}(R) \subseteq \mathrm{Z}(R)$, hence $2e \in \mathrm{Z}(R)$. As $u \in \mathrm{U}(R) \subseteq \mathrm{Z}(R)$, we have $2eu \in \mathrm{Z}(R)$ and $u-u^2 \in \mathrm{Z}(R)$. Moreover, $e-e^2 \in \mathrm{rad}(R) \subseteq \mathrm{Z}(R)$. Consequently, $x-x^2 = (e-e^2)-2eu+(u-u^2) \in \mathrm{Z}(R)$, as required. □

Remark 2.4. The classical commutativity theorems of Jacobson, Herstein, and Kaplansky made heavy use of subdirect product representations. If R is a unit-central ring, and $R/\mathrm{rad}(R)$ is a finite subdirect product of simple artinian rings, then R is semilocal, and by Theorem 2.3, R must be commutative. One might be tempted to try to extend this conclusion to the case where $R/\mathrm{rad}(R)$ is a subdirect product of an arbitrary set of simple artinian rings. Unfortunately, this generalization fails. If k is an infinite field, and $\{x_i : i \in I\}$ is an infinite set of noncommuting indeterminates, then the free algebra $R = k\langle\{x_i : i \in I\}\rangle$ is a noncommutative unit-central ring with $\mathrm{rad}(R) = (0)$, and by [1, Corollary 3], R can be represented as a subdirect product of simple artinian rings.

Example 2.5. Let

$$R = \begin{pmatrix} \mathbb{F}_2 & V \\ 0 & \mathbb{F}_2 \end{pmatrix}$$

where V is any nonzero $\mathbb{F}_2$-vector space. Then R is a noncommutative semiprimary ring with commuting units. Thus, in Lemma 2.1(iii) and Theorem 2.3 the unit-central hypothesis cannot be weakened to "commuting units."

If, however, we assume that R has no factor ring isomorphic to $\mathbb{F}_2$, then Theorem 2.3 can be extended to rings with commuting units. To prove this, we will make use of the following theorem, which occurred (with different terminology) as [21, Theorem 2.2]. (The "left suitable" condition in [21, Theorem 2.2] is equivalent to the ring being an exchange ring: see [21, Lemma 1.2] or [23, Theorem 2.1].)

Theorem 2.6 (Nicholson, Springer). *A semiprime exchange ring with commuting units is commutative.*

The following theorem strengthens a result of J. Han, [16, Theorem 2.9]. We note that Nicholson has a complementary result for semiperfect rings, [22, Corollary 1(1)].

Theorem 2.7. *Let R be a semiexchange ring with commuting units. If R has no factor ring isomorphic to $\mathbb{F}_2$, then every element of R is a sum of two units, and consequently R is commutative.*

Proof. Let $\overline{R} = R/\mathrm{rad}(R)$. As $\overline{R}$ is a semiprimitive exchange ring with commuting units, Theorem 2.6 implies $\overline{R}$ is commutative. In a commutative exchange ring with no factor ring isomorphic to $\mathbb{F}_2$, every element is the sum of two units. (See [6, Theorem 3]; the idea was already implicit in [13, Theorem 2].) Thus, every element of $\overline{R}$ is the sum of two units, whence every element of R is the sum of two units. Since R has commuting units, it is commutative. □

An element a of a ring R is a *von Neumann regular element* if there exists some $b \in R$ such that $a = aba$.

Theorem 2.8. *Let R be a ring with the property that for all $a, b \in \mathfrak{N}(R)$ we have $ab = ba$. (In particular, any ring with commuting units has this property.) Then R does not contain any nonzero nilpotent von Neumann regular element.*

Proof. Assume the contrary, that there exists some $a \in R$ such that $a = aba$ for some $b \in R$, and $a^n = 0 \neq a^{n-1}$ for some integer $n \geqslant 2$. Put $e = ab$.

The nilpotent elements a and $ea(1-e)$ commute, hence $aea(1-e) = 0$. From $a^2(1-e) = 0$ we obtain $a^2 = a^2e = a^3b$, whence $a^2 = a^{m+2}b^m$ for every $m \in \mathbb{N}$. Therefore $a^2 = 0$, which implies $a = ea(1-e)$. Hence

$$\begin{aligned} a &= \Big(ea(1-e)\Big)b\Big(ea(1-e)\Big) \\ &= \Big(ea(1-e)\Big)\Big((1-e)be\Big)\Big(ea(1-e)\Big) \\ &= \Big(ea(1-e)\Big)\Big(ea(1-e)\Big)\Big((1-e)be\Big) \\ &= 0, \end{aligned}$$

a contradiction. □

Using Theorem 2.8, we can recover the following special case of Theorem 2.6.

Corollary 2.9 (Nicholson, Springer). *Any von Neumann regular ring with commuting units is commutative.*

Proof. Let R be a von Neumann regular ring with commuting units. By Theorem 2.8, R is reduced and von Neumann regular, i.e. strongly regular. In a strongly regular ring every element is the product of a unit and a central idempotent. Hence R is commutative. □

Remark 2.10. It follows from Theorem 2.3 that any unit-central artinian ring is commutative. A unit-central noetherian ring, however, need not be commutative. For instance, the Weyl algebras over a field are noncommutative unit-central noetherian rings. For another example of this sort, let $A = k[t_1, t_2, \ldots, t_n]$ where k is a field and the t_i's are commuting indeterminates, and let σ be any nonidentity k-linear automorphism of A. Then the skew polynomial ring $R = A[x; \sigma]$ is a noncommutative unit-central noetherian ring.

We note in closing that commutativity theorems complementary to those in this section can be found in [15, §5].

3. Open problems

A ring R is called *right duo* if each right ideal of R is two-sided. We ask the following question.

Question 3.1. *Is every unit-central right duo ring commutative?*

Following Nicholson and M. F. Yousif in [24], we call a ring R *right principally injective,* or *right P-injective,* if for every $a \in R$, every right R-module homomorphism $aR \to R$ extends to a right R-module homomorphism $R \to R$. Since a right self-injective ring is an exchange ring, we know that every unit-central right self-injective ring is commutative. This suggests the following question.

Question 3.2. *Is every unit-central right principally-injective ring commutative?*

Nicholson and E. Sánchez-Campos [25] called an element a of a ring R a *right morphic element* if $R/aR \cong \mathrm{ann}_r^R(a)$ as right R-modules. A ring R is called a *right morphic ring* if every element of R is a right morphic element. Clearly every unit and idempotent in a ring is morphic. We ask the following question.

Question 3.3. *Is every unit-central right morphic ring commutative?*

A ring R is said to have *stable range* 1 if for all $a, b \in R$ such that $aR+bR = R$, there exists $y \in R$ such that $a + by$ is a unit. As every semilocal ring has stable range 1, and Theorem 2.3 shows that every unit-central semilocal ring is commutative, we ask the following.

Question 3.4. *Is every unit-central ring with stable range* 1 *commutative?*

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