UNIT SUM NUMBERS OF RIGHT SELF-INJECTIVE RINGS

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Dedicated to the memory of late Professor Irving Kaplansky

Abstract. In a recent paper [KS] we proved that every element of a right self-injective ring R is a sum of two units if and only if R has no factor ring isomorphic to $\mathbb{Z}_2$ and hence the unit sum number of a nonzero right self-injective ring is 2, ω or ∞. In this paper we characterize right self-injective rings with unit sum numbers ω and ∞. We prove that the unit sum number of a right self-injective ring R is ω if and only if R has a factor ring isomorphic to $\mathbb{Z}_2$ but no factor ring isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, and also in this case every element of R is a sum of either two or three units. It follows that the unit sum number of a right self-injective ring R is ∞ precisely when R has a factor ring isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. We also answer a question of Henriksen ([H], Question E, page 192) by giving a large class of regular right self-injective rings having the unit sum number ω in which not all non-invertible elements are sum of two units.

We shall consider associative rings with identity. Our modules will be unital right modules with endomorphisms acting on the left.

A ring R is said to have *n-sum property*, for a positive integer n, if its every element can be written as a sum of exactly n units of R. It is obvious that a ring having n-sum property also has k-sum property for every positive integer $k > n$. The *unit sum number* of a ring R, denoted by $\operatorname{usn}(R)$, is the least integer n, if it exists, such that R has the n-sum property. If R has an element which is not a sum of units then we set $\operatorname{usn}(R)$ to be ∞, and if every element of R is a sum of units but R does not have n-sum property for any n, then we set $\operatorname{usn}(R) = \omega$. Clearly, $\operatorname{usn}(R) = 1$ if and only if R has only one element. The unit sum number of a module M, denoted by $\operatorname{usn}(M)$, is the unit sum number of its endomorphism ring. The topic has been studied extensively (see [AV], [FS], [GMW], [GO], [GPS], [H], [Hi], [R], [U], [V], [Z]).

In 1954 Zelinsky [Z] proved that $\operatorname{usn}(V_D) = 2$, where V is a vector space over a division ring D, unless $\dim(V) = 1$ and $D = \mathbb{Z}_2$, in which case $\operatorname{usn}(V_D) = 3$. As $\operatorname{End}(V_D)$ is a (von Neumann) regular ring, in 1958 Skornyakov ([S], Problem 31, page 167) asked: *Is every element of a regular ring a sum of units?* This question was settled by Bergman (see [Ha]) in negative who gave an example of a regular directly-finite ring R in which 2 is invertible such that $\operatorname{usn}(R) = \infty$. The result of Zelinsky has also generated a considerable interest in the unit sum numbers of right self-injective rings as $\operatorname{End}(V_D)$ is right self-injective (see [U], [V], [R]). Recently, the authors in [KS] have characterized right self-injective rings with unit sum number

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2, thereby generalizing the result of Zelinsky. It is proved in [KS] that for a right self-injective ring R, $\mathrm{usn}(R) = 2$ if and only if the identity of R is a sum of two units, if and only if R has no factor ring isomorphic to $\mathbb{Z}_2$. So it is immediate that the unit sum number of a nonzero right self-injective ring is 2, ω or ∞. In this paper we prove that the right self-injective rings with unit sum number ω are precisely the rings which have a factor ring isomorphic to $\mathbb{Z}_2$ but no factor ring isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. It follows that a right self-injective ring has the unit sum number ∞ if and only if it has a factor ring isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. In ([H], Question E, page 192) Henriksen asked if there exists a regular ring R with $\mathrm{usn}(R) \neq \infty$ which has a non-invertible element that can't be written as a sum of two units. We answer this question by giving a large class of regular right self-injective rings having the unit sum number ω such that not all non-invertible elements are sum of two units. However, we prove that if R is a regular right self-injective ring with $\mathrm{usn}(R) \neq \infty$ then every non-invertible element in R is a sum of either two or three units.

The work depends heavily on the Type theory of regular right self-injective rings introduced by Kaplansky [K]. While writing the paper we received the sad news of the demise of Professor Kaplansky. We dedicate this paper to the memory of Professor Kaplansky.

For the terms not defined we refer the reader to [G]. The following two lemmas are inherent in [KS] and we include them here for the sake of completeness.

Lemma 1 *For any nonzero regular right self-injective ring R, usn$(M_2(R)) = 2$. In particular, for any purely infinite regular right self-injective ring R, usn$(R) = 2$.*

Proof. By ([AGMP], Corollary 2.6) every $A \in M_2(R)$ admits a diagonal reduction i.e., there exist invertible matrices P and Q in $M_2(R)$ such that PAQ is a diagonal matrix. Clearly PAQ and so A can be written as a sum of two units.

If R is purely infinite regular right self-injective ring, then $R_R \cong (R \oplus R)_R$ implying that, as rings, $R \cong M_2(R)$ and so the result follows from above. □

Using Lemma 1 we get the following

Lemma 2 *A nonzero regular right self-injective ring R has a ring decomposition $R = S \times T$ where usn$(S) = 1$ or 2 and T is an abelian regular right self-injective ring.*

Proof. By ([G], Proposition 10.21) $R \cong R_1 \times R_2$, where R_1 is either zero or purely infinite and R_2 is directly finite. By Lemma 1, $\mathrm{usn}(R_1) = 1$ or 2. By ([G], Theorem 10.22) $R_2 \cong R_3 \times R_4$, where R_3 is Type I_f and R_4 is Type II_f. Further, by ([G], Theorem 10.24), $R_3 \cong \prod M_n(S_i)$, where each S_i is an abelian regular right self-injective ring. Also, as R_4 has no nonzero abelian idempotents, there exists an idempotent $e \in R_4$ such that $(R_4)_{R_4} \cong (eR_4 \oplus eR_4)_{R_4}$ (see [G], Proposition 10.28), implying that, as rings, $R_4 \cong M_2(eR_4e)$. Thus $R \cong R_1 \times (\prod M_n(S_i)) \times M_2(eR_4e)$. We let $S = R_1 \times (\prod_{n>1} M_n(S_i)) \times M_2(eR_4e)$ and $T = \prod S_i$. By Lemma 1 $\mathrm{usn}(S) = 1$ or 2. □

Lemma 3 (Vámos [V], Proposition 19) *Let M_R be a nonsingular injective module. Then End(M_R) is a Boolean ring if and only if the identity endomorphism of no* *direct summand of M is a sum of two units.*

Lemma 4 ([KS], Theorem 1) *For a nonzero right self-injective ring R, the following conditions are equivalent:*

(1) usn(R)=2.

(2) The identity of R is a sum of two units.

(3) R has no factor ring isomorphic to $\mathbb{Z}_2$.

Lemma 5 *Let T be an abelian regular right self-injective ring. Then $T = T_1 \times T_2$ where usn$(T_1) = 1$ or 2 and T_2 is either zero or a Boolean ring.*

Proof. In view of Lemma 4 it is enough to show that T has a ring decomposition $T = T_1 \times T_2$, where T_2 is a Boolean ring and the identity in T_1 is a sum of two units in T_1. We shall prove this using a standard Zorn's lemma argument as used in ([V], Lemma 17). Let $\mathcal{T}$ be the set of all pairs of the form (A, u) where A is a submodule of T_T and u is an automorphism of A such that $I_A - u$ is also an automorphism of A, where I_A is the identity automorphism of A. Clearly $(0, 0) \in \mathcal{T}$. Then $\mathcal{T}$ has an obvious partial order i.e., $(A, u) \leq (A', u')$ if $A \subseteq A$ and u' agrees with u on A. Also $\mathcal{T}$ is easily seen to be inductive and so by Zorn's lemma $\mathcal{T}$ has a maximal element (T_1, v) say. If T_1 is not injective, then v extends to an automorphism v' of an injective hull $E(T_1)$ of T_1 in T_T, and so $(E(T_1), v') \in \mathcal{T}$ violating the maximality of (T_1, v). Thus T_1 is injective and so $T = T_1 \oplus T_2$ for some submodule T_2 of T_T. As (T_1, v) is a maximal element of $\mathcal{T}$, it is clear that the identity endomorphism of no direct summand of T_2 is a sum of two units. So, by Lemma 3, $End_T(T_2)$ is a Boolean ring. As every idempotent in T is central, the module decomposition $T_T = T_1 \oplus T_2$ gives us a ring decomposition $T = T_1 \times T_2$ with $End_T(T_2) \cong T_2$ a Boolean ring. Also the identity in T_1 is a sum of two units in T_1. □

We know that a right self-injective ring is right continuous (see [G]). The following example shows that Lemma 5 is not true even for commutative regular continuous rings. Let F be a field with $\mathbb{Z}_2$ its proper subfield. Set $F_n = F$ and $K_n = \mathbb{Z}_2$ for each positive integer n. Let

$$R = \{(x_n)_{\mathbb{N}} \in \prod_{\mathbb{N}} F_n : x_n \in K_n \text{ for all but finitely many } n\}.$$

Then R is a commutative continuous ring (see [G], Example 13.8) which clearly is regular. It is easy to see that the identity of R is not a sum of two units. An element $(x_n)_{\mathbb{N}}$ of R is an idempotent precisely when every component is either 0 or 1 and so it is clear that eR is not a Boolean ring for any idempotent $e \in R$. Also the element $(1, 0, 1, 0, ...)$ is not a sum of units in R implying that usn$(R) = \infty$.

The following result characterizes the regular right self-injective rings with various unit sum numbers.

Theorem 6 *The unit sum number of a nonzero regular right self-injective ring R is 2, ω or ∞. Moreover,*

(1) usn$(R) = 2$ if and only if R has no (nonzero) Boolean ring as a ring direct summand.

(2) usn$(R) = \omega$ *if and only if* R *has* $\mathbb{Z}_2$*, but no Boolean ring with more than two elements, as a ring direct summand. Moreover, in this case every non-invertible element of* R *is a sum of either two or three units.*

(3) usn$(R) = \infty$ *if and only if* R *has a Boolean ring with more than two elements as a ring direct summand.*

Proof. In view of Lemma 2 and Lemma 5, $R = R_1 \times B$ where $\text{usn}(R_1) = 1$ or 2 and B is a Boolean ring. It is clear that the unit sum number of a nonzero Boolean ring is ∞ unless it is isomorphic to $\mathbb{Z}_2$, in which case the unit sum number is ω. So (1) is immediate. If $\text{usn}(R) \neq 2$, then $B \neq 0$. Clearly, $\text{usn}(R) = \omega$ if and only if $B \cong \mathbb{Z}_2$ and $\text{usn}(R) = \infty$ if and only if B has more than two elements. Also if $\text{usn}(R) = \omega$, then $R \cong R_1 \times \mathbb{Z}_2$ and it is clear than any non-invertible element of R is a sum of either two or three units. □

In ([H], Question E, page 192) Henriksen asked if there is a regular ring, with every element sum of units, in which there are non-invertible elements that are not sum of two units. The following example answers this question.

Example 7 Let S be a nonzero regular right self-injective ring which does not have a factor ring isomorphic to $\mathbb{Z}_2$. For instance, take S to be any field other than $\mathbb{Z}_2$. Let $R = S \times \mathbb{Z}_2$. Clearly R is a regular right self-injective ring and, by Theorem 6, $\text{usn}(R) = \omega$. But the element $(0, 1)$ of R is a non-unit which can't be written as a sum of two units.

We are now ready to prove our main result.

Theorem 8 *The unit sum number of a nonzero right self-injective ring* R *is* 2*,* ω *or* ∞*. Moreover,*

(1) usn$(R) = 2$ *if and only if* R *has no factor ring isomorphic to* $\mathbb{Z}_2$*.*

(2) usn$(R) = \omega$ *if and only if* R *has a factor ring isomorphic to* $\mathbb{Z}_2$*, but has no factor ring isomorphic to* $\mathbb{Z}_2 \times \mathbb{Z}_2$*. In this case every non-invertible element of* R *is a sum of either two or three units.*

(3) usn$(R) = \infty$ *if and only if* R *has a factor ring isomorphic to* $\mathbb{Z}_2 \times \mathbb{Z}_2$*.*

Proof. We know that $R/J(R)$ is a nonzero regular right self-injective ring (see [U1]) and also it is clear that usn(R)=usn$(R/J(R))$. So the result follows from Theorem 6 and the fact that any Boolean ring with more than two elements has a factor ring isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. □

As defined above, the unit sum number of a module is the unit sum number of its endomorphism ring. So the unit sum number of any module, whose endomorphism ring modulo jacobson radical is right self-injective, is 2, ω or ∞. We list two such classes of modules below.

Corollary 9 *Let* M_R *be a quasi-injective or a flat cotorsion module. Then usn*(M) *is* 2*,* 3 *or* ∞*.*

Proof Let $E = End_R(M)$. Then $E/J(E)$ is a regular right self-injective (see [O] and [GH]). So the result follows from Theorem 8. □

We have seen that if R is a regular right self-injective ring with $\mathrm{usn}(R) \neq \infty$ then every non-invertible element is a sum of either two or three units. This makes us ask the following question.

Question 10 *Let R be a regular ring with $usn(R) \neq \infty$. Is every non-invertible element a sum of two or three units?*

The following question of Henriksen also seems to be open.

Question 11 (Henriksen [H], page 192) *Let R be a regular ring in which 2 is invertible such that $usn(R) \neq \infty$. Is every non-invertible element of R a sum of two units?*

In view of Theorem 6 the above question has positive answer for regular right self-injective rings.

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